ENS de Lyon Géométrie Algébrique
M1 2025-2026

TD9 : Lie-Kolchin and birational maps

Exercises with a  are algebraic geometry exercices which will be corrected during the exercise session, if time allows it. Exercises with a  are important exercices of commutative algebra.

**Exercise 1.** (*Lie-Kolchin theorem*) — Let G be a general group. A commutator in G is an element of the form $[f,g] := fgf^{-1}g^{-1}$, for some $f, g \in G$. Denote $D(G)$ or $D^1(G)$ the subgroup of G generated by all the commutators, denote $D^{r+1}(G) = D(D^r(G))$. G is said to be solvable if there is some r such that $D^r(G)$ is trivial.

1. Let G be a group. Show that $\forall r \geq 0$, $D^r(G)$ is a normal subgroup of G.

2. Let G be solvable and H be a quotient of G. Show that H is solvable.

3. Let k be algebraicly closed and $n \geq 1$. Let G be a subgroup of $\mathrm{GL}_n(k)$. Suppose that G is solvable and connected. We want to show that there is some $g \in \mathrm{GL}_n(k)$ such that the elements of gGg^{-1} are upper-triangular matrices.

a. Show that an abelian subgroup of $\mathrm{GL}_n(k)$ admits a stable line.

b. Show that $\forall n > 0, D^r(G)$ is connected. Deduce that there is a non-trivial normal connected abelian subgroup H of G. Denote $\mathcal{P}$ the set of (non-zero) eigenvectors for H, meaning that

$$\forall x \in \mathcal{P}, \forall h \in H, \exists \Lambda_x(h) \in k^\times, h \cdot x = \Lambda_x(h)x.$$

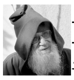
c. Show that $\mathcal{P}$ is stable by the action of G. Moreover, if $x \in \mathcal{P}$, express $\Lambda_{g(x)}(h)$. Deduce that the map $g \in G \mapsto \Lambda_{g(x)}(h) \in \mathbb{A}^1_k$ is constant.

d. Deduce that there is a space $0 \subsetneq V \subsetneq k^n$ that is stable by the action of G (you may separate discuss if G is abelian or not. If G is not abelian, what can one say about the determinant on H ?).

e. Show Lie-Kolchin's theorem.

4. Show the converse (you can do it for every field k) : denote $T_n(k)$ the group of upper triangular matrices, and show that every subgroup of $T_n(k)$ is solvable. Show that $T_n(k)$ is Zariski-connected.

5. What happens in question 1 if the connectedness condition is dropped ? And if the solvableness is dropped ?



Exercise 2. (*Being careful with products of varieties*) — Recall that the set $\mathbb{P}^n_k \times \mathbb{P}^m_k$ is not naturally a variety. For example, there is therefore no notion of morphisms or rational maps from this set. In TD7, we have shown that the Segre embedding

$$\rho : \begin{cases} \mathbb{P}^n_k \times \mathbb{P}^m_k & \to \mathbb{P}^{(n+1)(m+1)-1}_k \\ ([x_0, \dots, x_n], [y_0 : \dots : y_m]) & \mapsto [x_0 y_0 : x_0 y_1 : \dots : x_i y_i : \dots : x_n y_m]. \end{cases}$$

is a homeomorphism between its source and its image. Then the image of ρ, denoted by V, has a structure of an algebraic subset. More importantly, it has it a Zariski topology. This is not the the product topology, as we saw.

Denote by $(X_0 : \dots : X_n)$ the coordinates on $\mathbb{P}^n_k$, by $(Y_0 : \dots : Y_m)$ those on $\mathbb{P}^m_k$, and (Z_{ij}) those on $\mathbb{P}^{(n+1)(m+1)-1}_k$.

1. Show that V is a variety.

2. Describe its field of rational functions (in terms of some ring of functions of the coordinates $(X_0, \dots, X_n)$ and $(Y_0, \dots, Y_m)$).

Remark. Identifying $\mathbb{P}^n_k \times \mathbb{P}^m_k$ with this new structure, the Segre embedding becomes an isomorphism map (this is just transfering the structure). The only common thing between this variety and the product $\mathbb{P}^n_k \times \mathbb{P}^m_k$ is the set (we will still denote it like the product, most of the time).

Exercise 3. (*Some birational isomorphisms*) —

1. Let $U_0 = \{x_0 \neq 0\} \subset \mathbb{P}^n_k$, $V_0 = \{y_0 \neq 0\} \subset \mathbb{P}^m_k$ and $W_0 = \{z_0 \neq 0\} \subset \mathbb{P}^{n+m}_k$. Show that there is an isomorphism $U_0 \times V_0 \simeq W_0$ that you will describe in terms of projective coordinates. Deduce that $\mathbb{P}^n_k \times \mathbb{P}^m_k$ and $\mathbb{P}^{m+n}_k$ are birational.

2. Let $S = V_+(XY - ZW) \subset \mathbb{P}^3_k$, and $\Sigma := S \cap U_0 = V(Y - ZW) \subset \mathbb{A}^3_k$.

a. Show that S is the same as $\mathbb{P}^2_k$ (only use previous exercises) and that Σ is birational to $\mathbb{P}^2_k$.

b. Deduce that S contains two families of projective lines, each of the form $(L_t)_{t \in \mathbb{P}^1_k}$ verifying $\forall t, u, L_t \neq L_u \Rightarrow L_t \cap L_u = \emptyset$, and such that the intersection between one line from one family with one line from the other family is always a point.

c. Show that the Zariski topology on S is not the same as the product topology on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ (yes, again this kind of question).

Exercise 4. (*Regular locus of a projective hypersurface*) —

Assume k is of characteristic 0. Let $P \in k[X_0, \dots, X_n]$ be a homogeneous polynomial of degree d, and let $H = V_+(P) \subset \mathbb{P}^n_k$. We denote by ∇P the polynomial map $(\frac{\partial P}{\partial X_0}, \dots, \frac{\partial P}{\partial X_n}) : \mathbb{A}^{n+1}_k \to \mathbb{A}^{n+1}_k$. The regular subset H_{reg} of H is

$$H_{\text{reg}} = \{x \in H \mid \nabla P(x) \neq 0\}$$

Justify that this definition makes sense, and show that

$$H \setminus H_{\text{reg}} = \{x \in \mathbb{P}^n_k \mid \nabla P(x) = 0\}$$

Morever, if P is irreducible show that H_{reg} is open and dense in H.