TD9 : Lie-Kolchin and birational maps

Exercise 1. (*Lie-Kolchin theorem*) — Let G be a general group. A commutator in G is an element of the form $[f, g] := fgf^{-1}g^{-1}$, for some $f, g \in G$. Denote $D(G)$ or $D^1(G)$ the subgroup of G generated by all the commutators, denote $D^{r+1}(G) = D(D^r(G))$. G is said to be solvable if there is some r such that $D^r(G)$ is trivial.

1. Let G be a group. Show that $\forall r \geq 0$, $D^r(G)$ is a normal subgroup of G.

2. Let G be solvable and H be a quotient of G. Show that H is solvable.

3. Let k be algebraicly closed and $n \geq 1$. Let G be a subgroup of $\mathrm{GL}_n(k)$. Suppose that G is solvable and connected. We want to show that there is some $g \in \mathrm{GL}_n(k)$ such that the elements of gGg^{-1} are upper-triangular matrices.

a. Show that an abelian subgroup of $\mathrm{GL}_n(k)$ admits a stable line.

b. Show that $\forall n > 0, D^r(G)$ is connected. Deduce that there is a non-trivial normal connected abelian subgroup H of G. Denote $\mathcal{P}$ the set of (non-zero) eigenvectors for H, meaning that

$$\forall x \in \mathcal{P}, \forall h \in H, \exists \Lambda_x(h) \in k^\times, h \cdot x = \Lambda_x(h)x.$$

c. Show that $\mathcal{P}$ is stable by the action of G. Moreover, if $x \in \mathcal{P}$, express $\Lambda_{g(x)}(h)$. Deduce that the map $g \in G \mapsto \Lambda_{g(x)}(h) \in \mathbb{A}^1_k$ is constant.

d. Deduce that there is a space $0 \subsetneq V \subsetneq k^n$ that is stable by the action of G (you may separate discuss if G is abelian or not. If G is not abelian, what can one say about the determinant on H ?).

e. Show Lie-Kolchin's theorem.

4. Show the converse (you can do it for every field k) : denote $T_n(k)$ the group of upper triangular matrices, and show that every subgroup of $T_n(k)$ is solvable. Show that $T_n(k)$ is Zariski-connected.

5. What happens in question 1 if the connectedness condition is dropped ? And if the solvableness is dropped ?

Correction. 1. show that $D^n(G)$ is a characteristic subgroup of G, i.e., stable by every element of $\mathrm{Aut}(G)$. It is easy by induction on n.

2. The image of the $D(G)$ by the projection π is $D(\pi(G))$.

3. a. induction on n (either there are only homotheties, either one element is not, and restrict to one of its non trivial eigenspaces)

b. A continuous image of connected is connected and an union of connected sets having a common point is connected, hence $D^r(G)$ is connected. Now, G is solvable so the last non trivial $D^r(G)$ is abelian.

c. A finite set is not connected unless of cardinality 1. Continuous image of connected is connected. Λ has continuity properties.

d. Pick $x \in \mathcal{P}$, and consider $\mathrm{Vect}(G \cdot x)$. H acts on it by homotheties. H cannot be finite.

Exercise 2. (*Being careful with products of varieties*) — Recall that the set $\mathbb{P}^n_k \times \mathbb{P}^m_k$ is not naturally a variety. For example, there is therefore no notion of morphisms or rational maps from this set. In TD7, we have shown that the Segre embedding

$$\rho : \begin{cases} \mathbb{P}^n_k \times \mathbb{P}^m_k & \to \mathbb{P}^{(n+1)(m+1)-1}_k \\ ([x_0, \dots, x_n], [y_0 : \dots : y_m]) & \mapsto [x_0 y_0 : x_0 y_1 : \dots : x_i y_i : \dots : x_n y_m]. \end{cases}$$

is a homeomorphism between its source and its image. Then the image of ρ, denoted by V, has a structure of an algebraic subset. More importantly, it has it a Zariski topology. This is not the the product topology, as we saw.

Denote by $(X_0 : \dots : X_n)$ the coordinates on $\mathbb{P}^n_k$, by $(Y_0 : \dots : Y_m)$ those on $\mathbb{P}^m_k$, and (Z_{ij}) those on $\mathbb{P}^{(n+1)(m+1)-1}_k$.

1. Show that V is a variety.

2. Describe its field of rational functions (in terms of some ring of functions of the coordinates $(X_0, \dots, X_n)$ and $(Y_0, \dots, Y_m)$).

Remark. Identifying $\mathbb{P}^n_k \times \mathbb{P}^m_k$ with this new structure, the Segre embedding becomes an isomorphism map (this is just transfering the structure). The only common thing between this variety and the product $\mathbb{P}^n_k \times \mathbb{P}^m_k$ is the set (we will still denote it like the product, most of the time).

Correction. Pour l'irréductibilité : on pose $\mathfrak{a}$ l'idéal qu'on a vu au TD7, on a donc $V = \mathrm{Im}(\rho) = V(\mathfrak{a})$. Soit $\phi : k[Z_{ij}] \to k[X_i, Y_j]$ défini par $Z_{ij} \mapsto X_i Y_j$. On a clairement $\mathfrak{a} \subset \mathrm{Ker}(\phi)$. Donc $V(\mathrm{Ker}(\phi)) \subset V(\mathfrak{a})$. Si $z \in V(\mathfrak{a})$, alors x s'écrit $[x_i y_j, i, j]$ pour $[x_i, i] \in \mathbb{P}^n_k$ et $[y_j, j] \in \mathbb{P}^m_k$. Soit $f \in \mathrm{Ker}(\phi)$. Alors $f(X_i Y_j, i, j) = \phi(f) = 0$ donc

$$f(z) = f(x_i y_j, i, j) = f(X_i Y_j, i, j)(X_i = x_i, Y_j = y_j) = 0.$$

Donc $z \in V(\mathrm{Ker}(\phi))$. Donc $V(\mathrm{Ker}(\phi)) = V(\mathfrak{a})$.

To use the Nullstellensatz here, one has to check that $\mathrm{Ker}(\phi)$ is homogeneous, but it is easy because ϕ preserves homogeneity. More precisely, let $f \in \mathrm{Ker}(\phi)$, write $f = \sum_k f^{(k)}$, with $f^{(k)}$ being k-homogeneous in the variables (Z_{ij}). Then $\phi(f) = \sum_k \phi(f^{(k)})$ and it is clear that the $\phi(f^{(k)})$ are $2k$-homogeneous in the (X_i, Y_j). Hence $\forall k, \phi(f^{(k)}) = 0$ so $f^{(k)} \in \mathrm{Ker}(\phi)$ hence $\mathrm{Ker}(\phi)$ is homogeneous.

Remark : the morphism ϕ is natural because it look like ρ^*. Beware : it is not ρ^*. In fact, ρ^* is not even defined because ρ is not a morphism between algebraic sets (this is why we are doing this exercise...).

$\mathrm{Ker}(\phi)$ is prime as the kernel of a morphism towards an integral domain. Hence the the Nullstellensatz implies that V is irreducible.

We also get that $\mathrm{Ker}(\phi) = \sqrt{\mathfrak{a}}$. One can check that $\sqrt{\mathfrak{a}} = \mathfrak{a}$ (exercise). L'argument précédent identifie ces deux idéaux pour un corps k algébriquement clos. Il existe un argument beaucoup plus général (mais plus douloureux) :

https://math.stackexchange.com/questions/353846/hartshorne-problem-1-2-14-on-segre-embedding

For the field of rational functions : map $\frac{f}{g} \in k(V)$ to the element $\frac{f((X_i Y_j)_{ij})}{g((X_i Y_j)_{ij})}$ of the set of bi-homogeneous polynomials (homogenous in X and in Y). You have to check that it is a morphism of rings (this is not just evaluating polynomials from a polynomial ring : this ring is built from scratch, see the course notes!). This is injective because... it is a morphism of fields (check that the new ring is a field), or maybe just notice that the kernel of this morphism has something to do with ϕ and use that $V = V(\mathrm{Ker}(\phi))$. The surjectivity is not hard : the destination ring is generated by elements of the form $\frac{X_i}{X_j}$ and $\frac{Y_k}{Y_k}$ and you can recover the first ones using $\frac{Z_{il}}{Z_{jl}}$ and the other ones a similar way.

Exercise 3. (*Some birational isomorphisms*) —

1. Let $U_0 = \{x_0 \neq 0\} \subset \mathbb{P}^n_k$, $V_0 = \{y_0 \neq 0\} \subset \mathbb{P}^m_k$ and $W_0 = \{z_0 \neq 0\} \subset \mathbb{P}^{n+m}_k$. Show that there is an isomorphism $U_0 \times V_0 \simeq W_0$ that you will describe in terms of projective coordinates. Deduce that $\mathbb{P}^n_k \times \mathbb{P}^m_k$ and $\mathbb{P}^{m+n}_k$ are birational.

2. Let $S = V_+(XY - ZW) \subset \mathbb{P}^3_k$, and $\Sigma := S \cap U_0 = V(Y - ZW) \subset \mathbb{A}^3_k$.

a. Show that S is the same as $\mathbb{P}^2_k$ (only use previous exercises) and that Σ is birational to $\mathbb{P}^2_k$.

b. Deduce that S contains two families of projective lines, each of the form $(L_t)_{t\in\mathbb{P}^1_k}$ verifying $\forall t, u, L_t \neq L_u \Rightarrow L_t \cap L_u = \emptyset$, and such that the intersection between one line from one family with one line from the other family is always a point.

c. Show that the Zariski topology on S is not the same as the product topology on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ (yes, again this kind of question).

Correction. 1. Define $f(x_0, \dots, x_n, y_0, \dots, y_m) = (1 : \frac{x_1}{x_0} : \dots : \frac{x_n}{x_0} : \frac{y_1}{y_0} : \dots : \frac{y_m}{y_0})$ using previous exercise. f is the birational map you need.

2. a. this is not only birational but isomorphic by definition.

b. $\mathbb{P}^1 \times \mathbb{P}^1$ is the answser.

c. the line $x = y$ is closed S but show it is not in $\mathbb{P}^1 \times \mathbb{P}^1$. If it were closed, call it V, and then for all $x \notin V$, there is some elementary closed subset $V(f) \times \mathbb{P}^1$ or $\mathbb{P}^1 \times V(f)$ for f homogeneous that contains V and that does not contain x. Show that such an f should be constant.

Exercise 4. (*Regular locus of a projective hypersurface*) —

1. Assume k is of characteristic 0. Let $P \in k[X_1, \dots, X_n]$ be an irreducible polynomial, and let $H = V(P) \subset \mathbb{A}^n_k$. Let $1 \leqslant i \leqslant n$. Show that the following are equivalent :

1. $\frac{\partial P}{\partial X_i}$ vanishes on H.
2. $\frac{\partial P}{\partial X_i} = 0$.
3. $P \in k[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]$.

2. Give a counterexample when $\mathrm{car}(k) > 0$.

3. Assume k is of characteristic 0. Let $P \in k[X_0, \dots, X_n]$ be a homogeneous polynomial of degree d, and let $H = V_+(P) \subset \mathbb{P}^n_k$. We denote by ∇P the polynomial map $(\frac{\partial P}{\partial X_0}, \dots, \frac{\partial P}{\partial X_n}) : \mathbb{A}^{n+1}_k \to \mathbb{A}^{n+1}_k$. The regular subset H_{reg} of H is

$$H_{\mathrm{reg}} = \{x \in H \mid \nabla P(x) \neq 0\}$$

Justify that this definition makes sense, and show that

$$H \setminus H_{\mathrm{reg}} = \{x \in \mathbb{P}^n_k \mid \nabla P(x) = 0\}$$

Morever, if P is irreducible show that H_{reg} is open and dense in H.

Correction. Open is easy, dense is in the course notes (Reid for example. It suffices to show that it is non empty, use question 1). For the description of $H - H_{\mathrm{reg}}$, you just have to show that $\nabla P(x) = 0$ implies that $P(x) = 0$. The trick is that P is homogeneous so P can be recovered using $\sum_{i=1}^n X_i \frac{\partial P}{\partial X_i}$ (compute that).