ENS de Lyon
M1

Géométrie Algébrique
2025-2026

TD7 : Morphisms, projective geometry

Exercises with a  are algebraic geometry exercices which will be corrected during the exercise session, if time allows it. Exercises with a  are important exercices of commutative algebra.

Nothing too hard today, besides exercise 4.

**Exercise 1.** (*Why we like projective geometry*) — **1.** Let $F \subset \mathbb{P}^n(k)$ be Zariski-closed, for k being $\mathbb{R}$ or $\mathbb{C}$. Show that F is compact for the usual topology.

2. Show that the Zariski-closed and usual-topology-compacts of $\mathbb{A}^n(\mathbb{C})$ are the finite sets. Find a counter-example to this proposition for $\mathbb{A}^n(\mathbb{R})$.

**Exercise 2.** (*Examples of morphisms*) — **1.** Assume that $\mathrm{car}(k) = p$. Consider the map $\varphi : \mathbb{A}^n_k \to \mathbb{A}^n_k$ given by $\varphi(x_1, \ldots, x_n) = (x_1^p, \ldots, x_n^p)$. Show that φ is a bijective morphism and even an homeomorphism. Show that φ is not an isomorphism.

2. Let $C = V(x^3 - y^2) \subset \mathbb{A}^2_k$. Show that there is a bijective morphism $f : \mathbb{A}^1_k \to C$, but that such a morphism cannot be an isomorphism.

**Exercise 3.** (*The Segré embedding*) — Let k be a field. If $n, m \in \mathbb{N}$, consider the map

$$\rho : \begin{cases} \mathbb{P}^n_k \times \mathbb{P}^m_k & \to \mathbb{P}_k^{(n+1)(m+1)-1} \\ ([x_0, \ldots, x_n], [y_0 : \ldots : y_m]) & \mapsto [x_0 y_0 : x_0 y_1 : \ldots : x_i y_i : \ldots : x_n y_m]. \end{cases}$$

1. Show that ρ is well-defined, injective and closed (sends closed sets onto closed sets).

2. Is it clear that ρ is continuous ?

3. We define a new topology, the Zariski topology on $\mathbb{P}^n_k \times \mathbb{P}^m_k$, to be the topology for which the closed subset are the zero locus of bi-homogeneous polynomials, that is polynomials $P \in k[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$ which are homogeneous in $(X_0, \ldots, X_n)$ and in $(Y_0, \ldots, Y_m)$. You can imagine a similar definition for $\mathbb{P}^{n_1}_k \times \cdots \times \mathbb{P}^{n_m}_k$.

a. Show that the projection $\mathbb{P}^n_k \times \mathbb{P}^m_k \to \mathbb{P}^n_k$ is continuous.

b. Show, more precisely, that the product topology on $\mathbb{P}^n_k \times \mathbb{P}^m_k$ is stricly coarser than the Zariski topology (« coarse »=« grossière »= « has less open sets »).

c. Show that under this new topology, ρ is homeorphic between $\mathbb{P}^n_k \times \mathbb{P}^m_k$ and its image.

d. Deduce a precise answer to question 2 and explain why we needed to bring a new topology. This choice of topology becomes natural with scheme theory (I think it is because one prefers fibered products over direct products).

The following one will done in details next time. Don't do it now unless you feel bored. It may be a bit hard without hints.

Exercise 4. — Let G be a general group. A commutator in G is an element of the form $[f, g] := fgf^{-1}g^{-1}$, for some $f, g \in G$. Denote $D(G)$ or $D^1(G)$ the subgroup of G generated by all the commutators, denote $D^{k+1}(G) = D(D^k(G))$. G is said to be solvable if there is some k such that $D^k(G)$ is trivial.

1. Let k be algebraicly closed and $n \geq 1$. Let G be a Zariski-closed subgroup of $\mathrm{GL}_n(k)$. Suppose that G is solvable and connected. Show that there is some $g \in G$ such that the elements of gGg^{-1} are upper-triangular matrices.

2. Show the converse (you can do it for every field k) : denote $T_n(k)$ the group of upper triangular matrices, and show that every subgroup of $T_n(k)$ is solvable. Show that $T_n(k)$ is Zariski-connected.

3. What happens in question 1 if the connectedness condition is dropped ? And if the solvableness is dropped ?