

TD7 : MORPHISMS, PROJECTIVE GEOMETRY

Exercise 1. (*Why we like projective geometry*) — **1.** Let $F \subset \mathbb{P}^n(k)$ be Zariski-closed, for k being \mathbb{R} or \mathbb{C} . Show that F is compact for the usual topology.

2. Show that the Zariski-closed and usual-topology-compacts of $\mathbb{A}^n(\mathbb{C})$ are the finite sets. Find a counter-example to this proposition for $\mathbb{A}^n(\mathbb{R})$.

Correction. 1. $\mathbb{P}^n(k)$ is compact, and a closed set in a compact is compact also.
2. Do it for sets of the form $V(f)$ and you're good to go.



Exercise 2. (*Examples of morphisms*) — **1.** Assume that $\text{car}(k) = p$. Consider the map $\varphi : \mathbb{A}_k^n \rightarrow \mathbb{A}_k^n$ given by $\varphi(x_1, \dots, x_n) = (x_1^p, \dots, x_n^p)$. Show that φ is a bijective morphism and even an homeomorphism. Show that φ is not an isomorphism.

2. Let $C = V(x^3 - y^2) \subset \mathbb{A}_k^2$. Show that there is a bijective morphism $f : \mathbb{A}_k^1 \rightarrow C$, but that such a morphism cannot be an isomorphism.

Correction. 1. Well-defined, bijective, continuous are obvious. For homeomorphism, show that it's closed : the direct image of $V(f)$, for $f = \sum_I a_I X_I$ is $V(f')$ for $f' = \sum_I a_I^p X_I$. It's not an isomorphism because f^* is not surjective because of image $k[X_1^p, \dots, X_n^p] \subset k[X_1, \dots, X_n]$.

2. Show that $f(t) = (t^2, t^3)$ works but f^* is not an isomorphism. Because it would mean that there is some $P \in k[X, Y]$ such that $T = P(T^2, T^3)$. You can also show that the monic equation $T^2 - X \in k[C]$ has one solution in $\text{Frac}(k[C])$ but not $k[C]$, so $k[C]$ is not integrally closed.

Exercise 3. (*The Segre embedding*) — Let k be a field. If $n, m \in \mathbb{N}$, consider the map

$$\rho : \begin{cases} \mathbb{P}_k^n \times \mathbb{P}_k^m & \rightarrow \mathbb{P}_k^{(n+1)(m+1)-1} \\ ([x_0, \dots, x_n], [y_0 : \dots : y_m]) & \mapsto [x_0 y_0 : x_0 y_1 : \dots : x_i y_i : \dots : x_n y_m]. \end{cases}$$

1. Show that ρ is well-defined, injective and closed (sends closed sets onto closed sets).

2. Is it clear that ρ is continuous ?

3. We define a new topology, the Zariski topology on $\mathbb{P}_k^n \times \mathbb{P}_k^m$, to be the topology for which the closed subset are the zero locus of bi-homogeneous polynomials, that is polynomials $P \in k[X_0, \dots, X_n, Y_0, \dots, Y_m]$ which are homogeneous in (X_0, \dots, X_n) and in (Y_0, \dots, Y_m) . You can imagine a similar definition for $\mathbb{P}_k^{n_1} \times \dots \times \mathbb{P}_k^{n_m}$.

a. Show that the projection $\mathbb{P}_k^n \times \mathbb{P}_k^m \rightarrow \mathbb{P}_k^n$ is continuous.

b. Show, more precisely, that the product topology on $\mathbb{P}_k^n \times \mathbb{P}_k^m$ is strictly coarser than the Zariski topology (« coarse » = « grossière » = « has less open sets »).

c. Show that under this new topology, ρ is homeorphic between $\mathbb{P}_k^n \times \mathbb{P}_k^m$ and its image.

d. Deduce a precise answer to question 2 and explain why we needed to bring a new topology. This choice of topology becomes natural with scheme theory (it is because one prefers fibered products over direct products).

Correction. 1. Well-defined is just checking that there is some non-zero $x_i y_j$ in the mix. Injectivity is not hard, do it.

First show that the image is closed. Notice that, if $(X_{ij})_{ij} \in \mathbb{P}_k^{(n+1)(m+1)-1}$ is in the image, then $\forall i, j, k, l, X_{ij} X_{kl} = X_{il} X_{kj}$. Let $V := V(X_{ij} X_{kl} - X_{il} X_{kj} \mid i, j, k, l)$, let's show that $V = \text{Im}(\rho)$.

Let $(x_{ij})_{ij}$ be an element of V . There is some i_0, j_0 such that $x_{i_0, j_0} \neq 0$. Then, for i , put $\alpha_i = \frac{x_{i, j_0}}{x_{i_0, j_0}}$. For each i , denote v_i the vector $(x_{i, 0}, \dots, x_{i, m})$ in k^{m+1} . Then one has $\forall i, v_i = \alpha_i v_{i_0}$. Then the element $([\alpha_0 : \dots : \alpha_n], v_{i_0})$ in $\mathbb{P}_k^n \times \mathbb{P}_k^m$ is sent to (x_{ij}) by ρ .

For the closedness, show that for all homogeneous $g \in k[X_0, \dots, X_n]$, the set $\rho(V(g) \times \mathbb{P}_k^m)$ is closed in $\mathbb{P}_k^{(n+1)(m+1)-1}$. This is easy, it's because this set is the locus of (X_{ij}) such that $\forall 0 \leq j \leq m, g(X_{0j}, \dots, X_{nj}) = 0$. Then ρ is closed because every closed set can be written as an intersection of such closed sets (or closed sets of the form $\mathbb{P}_k^n \times \rho(V(g))$, which are the same thing), and because ρ is injective its direct image commutes with intersections.

2. No, because the product topology on $\mathbb{P}_k^n \times \mathbb{P}_k^m$ has nothing to do with equations using both x and y . More precisely, it is not continuous. We have to find a counter-example. Beware : don't try to study inverse images of singletons : those are singletons hence closed. Same with finite sets. The next question suggests us to study $V(g)$ for g made from X and Y intertwined, homogeneous in X and Y . But g should not be of the form $a(X)b(Y)$ because one would have $V(g) = V(a) \times \mathbb{P}_k^m \cup \mathbb{P}_k^n \times V(b)$. X and Y should be really intertwined.

Consider $V = V(Z_{01} - Z_{10}) \subset \mathbb{P}_k^{(n+1)(m+1)-1}$. Put $F = \rho^{-1}(V) = V(x_0 y_1 - x_1 y_0)$. It's clear that F is not the whole space $\mathbb{P}_k^n \times \mathbb{P}_k^m$. So, if F were closed (for the product topology), then F would be contained in some $V(g_1) \times \mathbb{P}_k^m \cup \mathbb{P}_k^n \times V(g_2)$, for g_1 homogeneous in X and g_2 homogeneous in Y , with $g_1 \neq 0$ and $g_2 \neq 0$.

Let $\alpha \in k^\times$. Either there is some $(y_2, \dots, y_m) \in k^{m-1}$ such that $g_2(1, \alpha, y_2, \dots, y_m) \neq 0$ (situation 1), either for all $(y_2, \dots, y_m) \in k^{m-1}$, we have $g_2(1, \alpha, y_2, \dots, y_m) = 0$ (situation 2). In situation 2, we have $g_2(1, \alpha, Y_2, \dots, Y_m) = 0$ because k is infinite.

In situation 1, we then have that for all $(x_2, \dots, x_n) \in k^{n-1}$, the point $([1 : \alpha : x_2 : \dots : x_n], [1 : \alpha : y_2 : \dots : y_m])$ is in F but not in $\mathbb{P}_k^n \times V(g_2)$ because $[1 : \alpha : y_2 : \dots : y_m]$ is not in $V(g_2)$. So we have $[1 : \alpha : x_2 : \dots : x_n] \in V(g_1)$, in conclusion, in situation 1, we have $\forall (x_2, \dots, x_n) \in k^{n-1}, g_1(1, \alpha, x_2, \dots, x_n) = 0$. Hence $g_1(1, \alpha, X_2, \dots, X_n) = 0$ because k is infinite.

In both situations, we have either g_1 or g_2 killed when their first two variables are replaced by $(1, \alpha)$. But there are infinitely many possible $\alpha \in k^\times$ so there is one situation that happens infinitely many times (pigeon-hole principle). So, WLOG, there are infinitely many $\alpha \in k^\times$ such that $g_1(1, \alpha, X_2, \dots, X_n) = 0$. Hence the coefficients of $g_1(1, X_1, X_2, \dots, X_n)$, which are polynomials in X_1 , should be 0 because

k is infinite. So $g_1(1, X_1, \dots, X_n) = 0$. g_1 is homogeneous so $g_1 = 0$. Impossible.

3. a. Show that the inverse image of a closed subset $V(g)$ is closed (easy), and deduce that the inverse image of every closed set is closed (because every closed is contained in a $V(g)$).

3. b. It is coarser because from question a we get that the identity map from $\mathbb{P}_k^n \times \mathbb{P}_k^m$ with product topology to $\mathbb{P}_k^n \times \mathbb{P}_k^m$ with the new topology is continuous (beware, it's not the identity map in the category of topological spaces. Just the identity map between sets). It is strictly coarser because, otherwise, this map would be a homeomorphism. It would follow that ρ is continuous and contradict question 2.

3.c show that this map is closed (similar proof than question 1) and continuous (use question 3.b).

The following one will be done in details next time. Don't do it now unless you feel bored. It may be a bit hard without hints.

Exercise 4. — Let G be a general group. A commutator in G is an element of the form $[f, g] := fgf^{-1}g^{-1}$, for some $f, g \in G$. Denote $D(G)$ or $D^1(G)$ the subgroup of G generated by all the commutators, denote $D^{k+1}(G) = D(D^k(G))$. G is said to be solvable if there is some k such that $D^k(G)$ is trivial.

1. Let k be algebraically closed and $n \geq 1$. Let G be a Zariski-closed subgroup of $\mathrm{GL}_n(k)$. Suppose that G is solvable and connected. Show that there is some $g \in G$ such that the elements of gGg^{-1} are upper-triangular matrices.

2. Show the converse (you can do it for every field k): denote $T_n(k)$ the group of upper triangular matrices, and show that every subgroup of $T_n(k)$ is solvable. Show that $T_n(k)$ is Zariski-connected.

3. What happens in question 1 if the connectedness condition is dropped? And if the solvability is dropped?