ENS de Lyon — Géométrie Algébrique
M1 — 2025-2026

TD6 : Bonus exercises about morphisms

Exercises with a  are algebraic geometry exercices which will be corrected during the exercise session, if time allows it. Exercises with a  are important exercices of commutative algebra.

Let k be an **algebraicly closed** field. We denote by $\mathbb{A}^n(k)$ the set k^n, it is the *affine space of dimension n*. The *ring of coordinates on C* $k[C]$ of a set C is defined as $k[X_1, \dots, X_n]/I(C)$, where $I(C)$ is the ideal of functions vanishing on C.

Exercise 1. — **1.** Assume that $\mathrm{car}(k) = p$. Consider the map $\varphi : \mathbb{A}^n_k \to \mathbb{A}^n_k$ given by $\varphi(x_1, \dots, x_n) = (x_1^p, \dots, x_n^p)$. Show that φ is a bijective morphism and even an homeomorphism. Show that φ is not an isomorphism.

2. Let $C = V(x^3 - y^2) \subset \mathbb{A}^2_k$. Show that there is a bijective morpshim $f : \mathbb{A}^1_k \to C$, but that such a morphism cannot be an isomorphism.

Exercise 2. (*Finite surjective morphisms*) — A morphism of algebraic sets $f : X \to Y$ is said to be *finite surjective* if f is of dense image and if $f^* : k[Y] \to k[X]$ makes $k[X]$ into a finite type $k[Y]$-module.

1. Show that f is finite surjective if and only if f^* is injective and for every $a \in k[X]$, there is a monic polynomial $P \in k[Y][T]$ such that $f^*(P)(a) = 0$.

2. Show that a finite surjective morphism is surjective and that its fibres are finite.

3. Let X be an algebraic subset of $\mathbb{A}^n_k$ and let G be a finite subgroup of automorphisms of X. Recall (*cf* TD1) that $k[X]^G := \{a \in k[X] \mid \forall g \in G \ g^*a = a\}$ is a finite type k algebra. Show that there exist an algebraic set X/G and a natural morphism $\pi : X \to X/G$ of algebraic sets such that $\pi^* : k[X/G] \to k[X]$ is the inclusion $k[X]^G \to k[X]$. Show that π is finite surjective and that the fibres of π are exaclty the orbits of the action of G on X.

Exercise 2 will have its correction on the portail des études.

The following exercises won't be corrected in class but some of you may feel the need of having fun doing them. You may want to do them in order.

Exercise 3. (*Algebraic Groups*) — An affine algebraic group is an algebraic set G such that there exist three morphisms of algebraic sets $m : G \times G \to G$, $e : \mathbb{A}^0_k \to G$ and $i : G \to G$ that have the usual axioms of multiplication, unit element and inverse in a group.

Let X be an algebraic set. We say that G acts on X if there is a morphism of algebraic sets $G \times X \to G$ satisfying the usual axioms of a group action. If $g \in G$ and $x \in X$, we denote by $g \cdot X$ the image of (g, x).

1. Show that the abstract group G acts on $k[X]$ by $(g \cdot f)(x) = f(g^{-1} \cdot x)$ for all $f \in k[X]$, $g \in G$ and $x \in X$.

2. Show that $\mathrm{GL}_n(k)$ is an affine algebraic group. Show that $\mathbb{A}_k^n$ is an affine algebraic group.

3. Let $W \subset k[X]$ be a finite dimensional k-vector space. Let V be the vector space spanned by the $g \cdot f$ for $f \in W$ and $g \in G$. Show that the dimension of V is finite, and that if one identifies $V = k^{\dim V}$, then the action of G on V is algebraic.

4. Show that there exist a finite dimensional k-vector space V with an algebraic action of V, and a closed immersion $\phi : X \to V$ such that $\phi(g \cdot x) = g \cdot \phi(x)$ for $g \in G$ and $x \in X$. Here, by closed immersion we mean that ϕ is a morphism of algebraic sets which is closed and for which the restriction $X \to \phi(X)$ is an isomorphism, that is ϕ identifies X with a closed algebraic subset of V.

5. By taking $G = X$, show that every affine algebraic group is a closed subgroup of $\mathrm{GL}_n(k)$ for some $n \in \mathbb{N}$. This is why affine algebraic groups are sometimes called linear algebraic groups.

Exercise 4. (*Hard*) — Let B be k algebra which is an integral domain. Let $A \subset B$ be a sub k-algebra such that B is a finite type A-algebra. Show that for any nonzero $b \in B$ there is some nonzero $a \in A$ such that any $f \in \mathrm{Hom}_{k-Alg}(A, k)$ such that $f(a) \neq 0$ can be extended to a $g \in \mathrm{Hom}_{k-Alg}(B, k)$ such that $g(b) \neq 0$. *(Hint : You could first do the case $B = A[u]$ for some $u \in B$.)*

Exercise 5. (*A theorem of Chevalley*) — **1.** Let X, Y be algebraic sets and let $f : X \to Y$ be a morphism with dense image. Show that if U is a nonempty open of X, then $f(U)$ contains a nonempty open of Y. (*Hint :You may use the previous exercise.*)

2. Deduce the following theorem of Chevalley : If $f : X \to Y$ is a morphism of algebraic sets, then $f(X)$ contains a open of $\overline{f(X)}$.

3. Let $f : H \to G$ be a morphism of affine algebraic groups (that is, a morphism of algebraic sets that commute with the multiplication and unit of the algebraic groups.)

a. Show that the kernel of f is an affine algebraic group which is closed in H.

b. Using Chevalley theorem, show that the image of f is closed in G, hence is an affine algebraic group. *(Hint : You may show that if G is an algebraic group and U, V are two dense opens of G, then $G = V \cdot U$).*