



TD5 : MORPHISMS AND IRREDUCIBILITY



Exercises with a  are algebraic geometry exercises which will be corrected during the exercise session, if time allows it. Exercises with a  are important exercises of commutative algebra.

Let k be a field. We denote by $\mathbb{A}^n(k)$ the set k^n , it is the *affine space of dimension n* . The *ring of coordinates on C* $k[C]$ of a set C is defined as $k[X_1, \dots, X_n]/I(C)$, where $I(C)$ is the ideal of functions vanishing on C .



Exercise 1. (*Some polynomial curves*) —

Let k be a field.

1. Let $C = \{(t, t^2, t^3) \mid t \in k\} \subset \mathbb{A}_k^3$. Show that it is Zariski closed, compute $I(C)$ and show that $k[C] \simeq k[T]$.
2. Let $C = \{t^2, t^3 \mid t \in k\} \subset \mathbb{A}_k^2$. Show that it is Zariski closed, and compute $I(C)$. Do we have $k[C] \simeq k[T]$?
3. k is now algebraically closed. Are those curves irreducible?



Exercise 2. (*Irreducibility*) —

Let k be a field.

1. Irreducibility and coordinate ring

- a. (if not done in class) Show that an affine set V is irreducible if and only if $k[V]$ is integral.
- b. Let k be algebraically closed. Let $f \in k[\mathbb{A}_k^n]$. Give a necessary and sufficient condition on f for $V(f)$ to be irreducible.
- c. Show that $f = Y^2 + X^2(X - 1)^2$ is irreducible in $\mathbb{R}[X, Y]$, but that $V(f)$ is not irreducible. (take time to convince yourself that it's not a paradox given previous questions)

2. Let $I \subset X$ an irreducible subset of a topological space X , and $f : X \rightarrow Y$ a continuous map. Show that $f(I)$ is irreducible.

k is now an algebraically closed field.

Some examples.

3. Give the irreducible components and their ideals of the following algebraic subset of \mathbb{A}_k^3 (k is algebraically closed) :
 - a. $V(XYZ, X^2 + Y^2 + Z^2)$,
 - b. $V(X^2 + Y^2 + Z^2 - 1, 3X^2 + Y^2 - Z^2 - 1)$,
 - c. $V(XY, YZ, Zx)$,

d. $V(X^2 + Y^2 + Z^2, Z^2 - XY)$, and

e. $V(X^2 - YZ, Y^2 - XZ)$.

4. Show that the irreducible components of $V(X^3 - YZ, Y^2 - XZ) \subset \mathbb{A}_k^3$ are $\{t^3, t^4, t^5 \mid t \in k\}$ and a line that you will give.

5. Let k be an algebraically closed field. What are the irreducible components of $\mathcal{P} = \{P \in \mathfrak{M}_n(k) \mid P^2 = P\}$?



Exercise 3. —

Let k be an algebraically closed field.

1. Let $X \subset \mathbb{A}_k^n$ and $Y \subset \mathbb{A}_k^m$ be two algebraic sets.

a. Show that a morphism $f : X \rightarrow Y$ has a dense image if and only if the morphism $f^* : k[Y] \rightarrow k[X]$ is injective. Give an example of a morphism with a dense image which is not surjective.

b. Show that a morphism $f : X \rightarrow Y$ is surjective if and only if for every maximal ideal \mathfrak{m} of $k[Y]$ we have $f^*(\mathfrak{m})k[X] \neq k[X]$.

2. Show that the image of a morphism $f : \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^2$ is closed. Hold on, this is not easy. Recall that we have a map $f^* : k[X, Y] \rightarrow k[T]$.

a. Show that its kernel I is a prime ideal of $k[X, Y]$. What happens if $I \cap k[X]$ or $I \cap k[Y]$ is non zero?

b. Suppose for the sequel that $I \cap k[X] = 0$. Show that I is principal, generated by some irreducible g .

c. (interlude) What did we just show about prime ideals of $k[X, Y]$? Say something about its geometric meaning when comparing the sets $\text{Spec}(k[X, Y]/I)$ and $V(I)$ (for I any ideal). Notice that you did not use the fact that k was algebraically closed.

d. Notice that $\text{Im}(f) \subset V(g)$. Use a previous question to show the converse (you may use some commutative algebra lemma, or maybe do some hard computations to make some determinant appear).

e. (bonus) Find an algorithm to determine an equation of the image of this morphism (use the resultant from another exercise sheet).

3. Let $f : \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^1$ be the map $(x, y) \mapsto xy$. Is the map closed? Open? Does it have dense image?



Exercise 4. (*Connexity and functions*) — Let k be an algebraically closed field. Show that an algebraic set $V \subset \mathbb{A}_k^n$ is connected if and only if $\Gamma(V)$ has no nontrivial idempotent. Recall that an idempotent of a ring A is a element $p \in A$ such that $p^2 = p$, and that trivial idempotents are 0 and 1.