TD5 : Morphisms and irreducibility

Let k be a field. We denote by $\mathbb{A}^n(k)$ the set k^n, it is the *affine space of dimension n*. The *ring of coordinates on C* $k[C]$ of a set C is defined as $k[X_1, \dots, X_n]/I(C)$, where $I(C)$ is the ideal of functions vanishing on C.

Exercise 1. (*Some polynomial curves*) —

Let k be a field.

1. Let $C = \{(t, t^2, t^3) \mid t \in k\} \subset \mathbb{A}^3_k$. Show that it is Zariski closed, compute $I(C)$ and show that $k[C] \simeq k[T]$.

Correction.

Notice that $\forall t \in k$, $(x, y, z) = (t, t^2, t^3)$ implies that $x^3 = z$ and $y = x^2$. So it's trivial that $C \subset V(X^3 - Z, Y - X^2)$. The converse is not hard because of « $t = x$ ».

Remark. If you can't show the converse, maybe it's because it's ok to miss some points (some closed sets can't be completely parametrized), or maybe, it's because you had not all the equations you need !

The function ring is then $k[X, Y, Z]/(X^3 - Z, Y - X^2)$ but its the same as $k[X]$, because, in the quotient, one has $Y = X^2 \in k[X]$ and $Z = X^3 \in k[X]$. You could just use the isomorphism given by $X \mapsto T$, $Y \mapsto T^2$, $Z \mapsto T^3$.

2. Let $C = \{t^2, t^3 \mid t \in k\} \subset \mathbb{A}^2_k$. Show that it is Zariski closed, and compute $I(C)$. Do we have $k[C] \simeq k[T]$?

Correction.

$\forall t$, $(x, y) = (t^2, t^3)$ implies that $x^3 = t^6 = y^2$. Hence $C \subset V(X^3 - Y^2) := V$. To show the converse, let $(x, y) \in V$. We want t such that $(t^2, t^3) = (x, y)$. But we have $x = (y/x)^2$ (it's trivial if $x = 0$). So you're good to go with $t = y/x$ or $-y/x$. The function ring is now $k[X, Y]/(X^3 - Y^2)$.

We have then $X^3 = Y^2$ in this ring, this *rings* a bell : it should not be factorial. But hey, it's not that easy (X and Y are not even primes, check it by quotienting the ring by them and find nilpotents in the quotient to conclude it's not a domain).

Let's define $\phi : k[X, Y] \to k[T]$ by $\phi(X) = T^2$ and $\phi(Y) = T^3$. Notice that $(X^3 - Y^2) \subset \mathrm{Ker}(\phi)$. Let $Q \in \mathrm{Ker}(\phi)$. Do an euclidean division $Q = (Y^2 - X^3)E + F$, $\deg_Y(F) \leq 1$ so $F(X, Y) = YA(X) + B(X)$. Evaluate this expression with ϕ : $0 = T^3A(T^2) + B(T^2)$. But $B(T^2)$ is a sum of even monomials and $T^3A(T^2)$ is a sum of odd monomials. So $A = B = 0$. So $F = 0$ so $Q \in (X^3 - Y^2)$. So $(X^3 - Y^2) = \mathrm{Ker}(\phi)$. So $\phi : k[C] \to k[T]$ is injective.

Check that $\mathrm{Im}(\phi)$ is $A := \mathrm{Vect}_k(1, T^2, T^3, T^4, \dots) \subset k[T]$ (just no T). This is still not a proof of why $k[C]$ is not $k[T]$: there may be some hidden isomorphism. But it turns out that T^3 has no prime decomposition in A (how to guess it ? just notice that T^2 is prime in A, and look at T^3). Let $T^3 = PQ$, with $P, Q \in A$, be a decomposition. This decomposition also lives in $k[T]$ through $A \hookrightarrow k[T]$. Then, assuming it's a non trivial decomposition, we would have $P = T^2$ and $Q = T$. Impossible. Thus T^3

should be prime in A. Well yes, but actually no : A/T^3 contains T^2, which is non zero but of square zero.

Conclusion : $k[C]$ is a nice-looking domain, perharps a bit cheeky because not factorial.

3. k is now algebraicly closed. Are those curves irreducible ?

Correction.

For the first one, we actually have that C is isomorphic to $\mathbb{A}^1$ via $(x, y, z) \mapsto x$. It's hence homeomorphic hence irreducible whenever $\mathbb{A}^1_k$ is (true whenever k is infinite. We don't use here the fact that it's algebraicly closed). For the second one, let's use the fact that k is algebraicly closed. Show that $X^3 - Y^2$ is irreducible (the usual way, or read above because we just proved that the quotient is a domain), and conclude that it's also irreducible.

Exercise 2. (*Irreducibility*) —

Let k be a field.

1. Irreducibility and coordinate ring

a. (if not done in class) Show that an affine set V is irreducible if and only if $k[V]$ is integral.

Correction. Suppose that $k[V]$ is integral. Suppose that $V = V_1 \cup V_2$, where V_i are closed in V but not equal to V. Say $V_i = V(I_i) \cap V$, where I_i are ideals. So if $(f, g) \in I_1 - \{0\} \times I_2 - \{0\}$, then $\forall x \in V$, we have either $x \in V_1$, either $x \in V_2$, but $fg(x) = 0$. So $fg = 0$ on V so by integrity $f = 0$ or $g = 0$, impossible so I_1 or I_2 has to be 0 so V_1 or V_2 has to be V.

Conversely, if V is irreducible, then suppose that $fg = 0$, then $V = (V \cap V(f)) \cup (V \cap V(g))$, but V is irreducible so say $V \cap V(f) = V$ so $f = 0$ in $k[V]$.

b. Let k be algebraicly closed. Let $f \in k[\mathbb{A}^n_k]$. Give a necessary and sufficient condition on f for $V(f)$ to be irreducible.

Correction. If f is a power of an irreducible p, then $\sqrt{(f)} = (p)$ which is prime. If $\sqrt{(f)}$ is prime, decompose $f = \prod_i p_i$ as a product of primes so $f \in \sqrt{(f)}$ gives that some p_i should be in $\sqrt{(f)}$. Hence, for some $k > 0$, $f \mid p_i^k$ so f is a power of a prime.

c. Show that $f = Y^2 + X^2(X-1)^2$ is irreducible in $\mathbb{R}[X, Y]$, but that $V(f)$ is not irreducible. (take time to convince yourself that it's not a paradox given previous questions)

2. Let $I \subset X$ an irreducible subset of a topological space X, and $f : X \to Y$ a continous map. Show that $f(I)$ is irreducible.

Correction. Write using the definition.

k is now an algebraically closed field.

Some examples.

3. Give the irreducible components and their ideals of the following algebraic subset of $\mathbb{A}^3_k$ (k is algebraicly closed) :

a. $V(XYZ, X^2 + Y^2 + Z^2)$,

Correction.

The method : find some irreducible closed subsets that cover your set, then get rid of the ones included in others. This will be automatically the unique decomposition.

Notice that the equation XYZ implies 3 components. Let's study for an example the one with $X = 0$. Let i be such that $i^2 = -1$ (k is algebraicly closed). Notice that $X^2 + Y^2 + Z^2$ becomes simply $(Y - iZ)(Y + iZ)$. You thus have components of the form $V(X = 0, Y - iZ)$, which are lines so are irreducible (do a drawing, and prove everything of that, it's important that you do it yourself). You will find in total an union of 6 lines.

b. $V(X^2 + Y^2 + Z^2 - 1, 3X^2 + Y^2 - Z^2 - 1)$,

c. $V(XY, YZ, Zx)$,

d. $V(X^2 + Y^2 + Z^2, Z^2 - XY)$, and

e. $V(X^2 - YZ, Y^2 - XZ)$.

4. Show that the irreducible components of $V(X^3 - YZ, Y^2 - XZ) \subset \mathbb{A}_k^3$ are $\{t^3, t^4, t^5 \mid t \in k\}$ and a line that you will give.

5. Let k be an algebraically closed field. What are the irreducible components of $\mathcal{P} = \{P \in \mathfrak{M}_n(k) \mid P^2 = P\}$?

Correction.

You guessed it : they are the $\mathcal{P}_k$, the sets of projectors of rank k. They are closed in $\mathcal{P}$ because they are given by the equation $\operatorname{tr} M = k$. They are irreducible because they are a direct image of $\mathrm{GL}_n(k)$. You heard it right : it's simply because all projectors of rank k are conjugated over $\mathrm{GL}_n(k)$. So if $p \in \mathcal{P}$, then $g \in \mathrm{GL}_n(k) \mapsto gpg^{-1} \in \mathcal{P}_k$ is a surjection. You should be happy : it's a continuous image of something irreducible. *But why continuous ? Why irreducible ?*

Lemma. If we see $\mathrm{GL}_n(k)$ as a subset of $\mathbb{A}_k^{n^2}$, then it's it's irreducible : it's an open set in an irreducible set (this is true because k is infinite. Actually, k being algebraicly closed is useless here).

Beware : g^{-1} is polynomial in g and in the inverse of $\det(g)$ so it's a rational function. You should know how to prove that a rational function is continuous on its set of definition. Try it before reading.

Theorem. Let now $\phi : U \to \mathbb{A}_k^n$ be a rational function which is defined everywhere on the affine set U. Let's show that ϕ is continous. Write $\phi = (\frac{P_1}{Q_1}, \dots, \frac{P_n}{Q_n})$, P_i, Q_i polynomials, such the Q_i never vanish on U (it's possible by construction of ϕ). Take a basic closed set $V(f) \subset \mathbb{A}_k^n$ (f is a polynomial). Thus $\phi^{-1}(V(f))$ is the set of all $x \in U$ such that $f \circ \phi(x) = 0$. Write $f = \sum_I a_I \prod_i X_i$ (lazy way of writing down a multivariate polynomial). Then $\forall x \in U$, $f \circ \phi(x) = \sum_I a_I \prod_i P_i/Q_i$ so there should be some k great enough such that $(Q_1 \dots Q_n)^k \cdot (f \circ \phi)$ is a polynomial. The Q_i never vanish so we can describe $\phi^{-1}(V(f))$ as the zero locus of the polynomial $(Q_1 \dots Q_n)^k \cdot (f \circ \phi)$, hence it's closed. Hence ϕ is continuous. $\square$

Remark 1. The 2 ways of describing $\mathrm{GL}_n(k)$ are then homeomorphic for the Zariski topology.

Remark 2. The 2 ways of describing $\mathrm{GL}_n(k)$ may NOT be homeomorphic for other topologies. If k is a topological ring, then k^{n^2} and k^{n^2+1} have natural topology and it's natural to give $\mathrm{GL}_n(k)$ a topology as a subset of one of these spaces. You may NOT get the same topology, because *the inversion in a topological ring has no reason to be continuous.*

Exercise 3. —

Let k be an algebraicly closed field.

1. Let $X \subset \mathbb{A}_k^n$ and $Y \subset \mathbb{A}_k^m$ be two algebraic sets.

a. Show that a morphism $f : X \to Y$ has a dense image if and only if the morphism $f^* : k[Y] \to k[X]$ is injective. Give an example of a morphism with a dense image which is not surjective.

Correction. f^* is injective $\Leftrightarrow \forall g \neq 0, g \circ f \neq 0 \Leftrightarrow \forall g \neq 0, \mathrm{Im}(f) \cap U(g) \neq$. This is equivalent to saying that $\mathrm{Im}(f)$ is dense in Y because the sets of the form $U(g) = V(g)^c$ are a basis of open sets for the Zariski topology.

b. Show that a morphism $f : X \to Y$ is surjective if and only if for every maximal ideal $\mathfrak{m}$ of $k[Y]$ we have $f^*(\mathfrak{m})k[X] \neq k[X]$.

Correction. Suppose that f is surjective. Then if $\mathfrak{m}$ is maximal in $k[Y]$, then it corresponds to a point $y \in Y$, which is reached by f so there exists $x \in X$ such that $f(x) = y$. Suppose that $f^*(\mathfrak{m})k[X] = k[X]$, so it contains 1, so we can write $1 = \sum_i f^*(g_i)h_i$, $h_i \in k[X]$ and $g_i \in k[Y]$. Evaluate at $x : 1 = \sum_i g_i \circ f(x)h_i(x) = \sum_i g_i(y)h_i(x)$. Thus, the polynomial $\sum_i g_i(Y)h_i(x) \in k[Y]$, which is in $\mathfrak{m}_y$, is 1 when evaluated at y : contradiction [(i)].

Let's prove the converse : let's show that f is surjective assuming the given condition. Let $y \in Y$. Let $\mathfrak{m}_y$ the corresponding maximal ideal of $k[Y]$. Then $\mathfrak{m}_y k[X] \neq k[X]$ so it's contained in some maximal ideal $\mathfrak{n}$ of $k[X]$. By Nullstellensatz, $\mathfrak{n}$ is some $\mathfrak{n}_x$ for $x \in X$. Hence $\mathfrak{m}_y \subset (f^*)^{-1}(\mathfrak{m}_y k[X]) \subset (f^*)^{-1}(\mathfrak{n}_x)$ [(ii)]. Hence $\forall 1 \leq i \leq n$, $f^*(Y_i - y_i) \in \mathfrak{n}_x$ hence $f^*(Y_i - y_i)$ vanishes at x. Thus, by checking coordinate-wise, $f(x) = y$.

2. Show that the image of a morphism $f : \mathbb{A}_k^1 \to \mathbb{A}_k^2$ is closed. Hold on, this is not easy. Recall that we have a map $f^* : k[X,Y] \to k[T]$.

a. Show that its kernel I is a prime ideal of $k[X,Y]$. What happens if $I \cap k[X]$ or $I \cap k[Y]$ is non zero ?

Correction.

I is the kernel of a morphism towards a domain so it's prime (do it if it's not clear for you). If $I \cap k[X]$ is not 0, then it's of the form (g) for $g \in k[X] - \{0\}$ (note : g should be irreducible. Do it if not sure.). Let x be a root of f. Put I_x to be the ideal of $k[Y]$ obtained by evaluating elements of I at $X = x$ (show that it's an ideal, nothing here makes it trivial). Then $\mathrm{Im}(f)$ is just the finite union of the $V(X - x, I_x)$, hence closed. □

b. Suppose for the sequel that $I \cap k[X] = 0$. Show that I is principal, generated by some irreducible g.

Correction. Let J be the ideal of $k(X)[Y]$ generated by I. J is principal generated by some g, that we multiply by some $k[X]$ to force it into $k[X,Y]$, and you even get $g \in I$. Let's show that $I = (g)$. Let $f \in I$. Then we can write $f = gh$ for some $h \in k(X)[Y]$. Hence $\deg_Y f \geq \deg_Y g$, so g is of minimal degree in Y (excepted $0 \in I$). Perform an euclidean division $g = fq + r$, $q, r \in k(X)[Y]$, $\deg_Y(r) < \deg_Y(f)$. Chase

(i). Did we use the fact that k is algebraicly closed here ?

(ii). Remark : $(f^*)^{-1}(\mathfrak{n}_x)$ does not contain 1 so it's proper. By maximality, $\mathfrak{m}_y = (f^*)^{-1}(\mathfrak{n}_x)$. You may notice that this fact is totally useless here.

denominators in $k[X]$, so there is some $p(x) \in k[X]$ such that $p(x)g = fqp(x) + rp(x)$ but now $rp(x) \in I$. But $\deg_Y(rp(x))$ is too low so $r = 0$. Hence $f \in (g)$. Hence $I = (g)$. □

c. (interlude) What did we just show about prime ideals of $k[X, Y]$? Say something about its geometric meaning when comparing the sets $\mathrm{Spec}(k[X, Y]/I)$ and $V(I)$ (for I any ideal). Notice that you did not use the fact that k was algebraicly closed.

d. Notice that $\mathrm{Im}(f) \subset V(g)$. Use a previous question to show the converse (you may use some commutative algebra lemma, or maybe do some hard computations to make some determinant appear).

Correction. This question is hard. Let $\mathfrak{m}$ a maximal ideal of $k[X, Y]/(g)$, let's show that $k[T]f^*(\mathfrak{m}) \neq k[T]$. First, let's show that the morphism $f^* : k[X, Y]/(g) \to k[T]$ is integral, meaning that it gives $k[T]$ a structure of integral $k[X, Y]/(g)$-algebra, hence finite-type module). This notion of morphism is very closely linked to things being surjective on the geometry side. It suffices to show that T is integral. f is not constant so, when writing $f = (f_1, f_2)$, f_1 or f_2 is a non-constant polynomial in $k[T]$. Suppose it's f_1. Then $f_1(T) - f^*(X) = 0$. Hence, if you write $f_1 = \sum_i a_i T^i$, $a_i \in k$, with leading coefficient $a \in k^\times$ then the monic polynomial $a^{-1}\left(\sum_i a_i U^i - f^*(X)\right) \in k[X, Y]/(g)[U]$ (U is a formal indeterminate) vanishes at T. Hence T is integral on $k[X, Y]/(g)$, hence $k[T]$ is a finite-type $k[X, Y]/(g)$-module.

$\mathfrak{m}$ is an ideal of $k[X, Y]/(g)$ and $k[T]$ is a finite-type $k[X, Y]/(g)$-module. Hence, if we had $k[T]f^*(\mathfrak{m}) \neq k[T]$, then Nakayama's lemma would imply the existence of some $x \in \mathfrak{m}$ such that $(1 + x)k[T] = 0$. Hence $x = -1$ is invertible, impossible.

Remark. I'll upload Robin Carlier's notes about this proof, which includes an explicit proof (without Nakayama), and makes it clear why one talks about "determinant trick".

□

e. (bonus) Find an algorithm to determine an equation of the image of this morphism (use the resultant from another exercise sheet).

3. Let $f : \mathbb{A}^2_k \to \mathbb{A}^1_k$ be the map $(x, y) \mapsto xy$. Is the map closed ? Open ? Does it have dense image ?

Correction. This map is surjective.

Exercise 4. (*Connexity and functions*) — Let k be an algebraically closed field. Show that an algebraic set $V \subset \mathbb{A}^n_k$ is connected if and only if $\Gamma(V)$ has no nontrivial idempotent. *Recall that an idempotent of a ring A is a element $p \in A$ such that $p^2 = p$, and that trivial idempotents are* 0 *and* 1.