TD4 : The Zariski Topology

Exercises with a  are algebraic geometry exercices which will be corrected during the exercise session, if time allows it. Exercises with a  are important exercices of commutative algebra.

Let k be a field. We denote by $\mathbb{A}^n(k)$ the set k^n, it is the *affine space of dimension* n. The *ring of functions* $\Gamma(C)$ of a closed set C is defined as $k[X_1, \dots, X_n]/I(C)$, where $I(C)$ is the ideal of functions vanishing on C.

Exercise 1. — **1.** Is the topology on $\mathbb{A}_k^{m+n}$ the product topology on $\mathbb{A}_k^n \times \mathbb{A}_k^m$ (do it for $n = m = 1$, the same idea will work in general case) ?

2. Prove that a polynomial map $f : \mathbb{A}^n(k) \to \mathbb{A}^m(k)$ is continuous. Can we say the same for a polynomial map $f : \mathbb{A}^n(k) \to \mathbb{A}^m(k) \times \mathbb{A}^r(k)$? And for a polynomial map $f : \mathbb{A}^n(k) \times \mathbb{A}^r(k) \to \mathbb{A}^m(k)$?

Exercise 2. (*Zariski topology on sets of matrices*) —
We identify $\mathfrak{M}_n(k)$ and $\mathbb{A}_k^{n^2}$.

1. Show that $\mathrm{GL}_n(k) \subset \mathfrak{M}_n(k)$ is open and that $\mathrm{SL}_n(k) \subset \mathfrak{M}_n(k)$ is closed.

2. Find an equation that realises $\mathrm{GL}_n(k)$ as a closed subset of $\mathbb{A}_k^{n^2+1}$.

3. Show that $\mathrm{SO}(n) \subset \mathfrak{M}_n(k)$ is closed.

4. Show that the set of nilpotent matrices of $\mathfrak{M}_n(k)$ is closed

5. Show that the set of matrices with n distinct eigenvalues (in an algebraic closure) is open (construct some symmetric polynomial with the roots and call it, maybe, « discriminant ». You should use another exercise about symmetryc polynomials).

Exercise 3. (*Density of Zariski open subsets*) — Let k be a field.

1. Let $f \in k[X_1, \dots, X_n]$. Assume that f vanishes on k^n. Is f the zero polynomial ?

2. Let $f \in k[X_1, \dots, X_n]$. Let $g \in k[X_1, \dots, X_n]$ be another polynomial. Let $D(g) = \{(x_1, \dots, x_n) \in k^n \mid g(x_1, \dots, x_n) \neq 0\}$. Assume that f vanishes on $D(g)$. Is f the zero polynomial ?

3. Let R be any commutative ring and let $M \in \mathfrak{M}_n(R)$ be a square matrix. Denote by χ_M the characteristic polynomial of M. Using the above give a new proof that $\chi_M(M) = 0$ (do it for integral domains using the previous exercise first, then use this case for the general case).

Exercise 4. (*Some polynomial curves*) —

1. Let $C = \{(t, t^2, t^3) \mid t \in k\} \subset \mathbb{A}^3_k$. Show that it is Zariski closed, compute $I(C)$ and show that $\Gamma(C) \simeq k[T]$.

2. Let $C = \{t^2, t^3 \mid t \in k\} \subset \mathbb{A}^2_k$. Show that it is Zariski closed, and compute $I(C)$. Do we have $\Gamma(C) \simeq k[T]$?

Exercise 5. — Let k be an algebraically closed field. Show that an algebraic set $V \subset \mathbb{A}^n_k$ is connected if and only if $\Gamma(V)$ has no nontrivial idempotent. *Recall that an idempotent of a ring A is a element $p \in A$ such that $p^2 = p$, and that trivial idempotents are* 0 *and* 1.

Exercise 6. (*Symmetric Polynomials Theorem, Artin's anti-calculation proof*) — Let k be a field and $n \geq 1$. Put $L = k(X_1, \ldots, X_n)$ and $K = k(\Sigma_1, \ldots, \Sigma_n)$, where

$$\forall r, \Sigma_r = \sum_{1 \leq k_1 < \cdots < k_r \leq n} X_{i_1} \ldots X_{i_r}$$

are the elementary symmetric polynomials.

1. Show that L is the field of decomposition of some separable polynomial over K. (a polynomial is separable if it's coprime with its derivative).

2. Deduce that L/K is Galois and compute $\mathrm{Gal}(L/K)$.

3. Prove that $k[X_1, \ldots, X_n] \cap k(\Sigma_1, \ldots, \Sigma_n) = k[\Sigma_1, \ldots, \Sigma_n]$ (hint. show that one of those rings is integrally closed).

4. Prove the following theorem : let $P \in k[X_1, \ldots, X_n]$ be a symmetric polynomial, meaning that $\forall \sigma \in \mathfrak{S}_n$, $P(X_1, \ldots, X_n) = P(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$, then P can be written as a polynomial in the $\Sigma_1, \ldots, \Sigma_n$ with coefficients in k.

There exists a computational proof of this theorem that allows you to implement it on a computer. You *shall* see this proof in Agrégation.

Exercise 7. (*Zariski topology and classical topology*) —

1. Show that a closed subset of $\mathbb{A}^n(\mathbb{C})$ is closed for the usual topology of $\mathbb{C}^n$ and that an open of $\mathbb{A}^n(\mathbb{C})$ is dense in $\mathbb{C}^n$.

2. Show that the set $\{(t, e^t) \mid t \in \mathbb{C}\}$ is dense in $\mathbb{A}^2(\mathbb{C})$. Is it an algebraic set ?

3. Let X be a closed Zariski subset of $\mathbb{A}^n(\mathbb{C})$. Show that if X with its classical topology is connected, then X with its Zariksi topology is also connected. *(The converse is also true, but much harder, this would take us too far.)*

Exercise 8. (*A more difficult exercise*) — Let X the set of nilpotent matrices of $\mathfrak{M}_n(k)$, with k algebraically closed. We have shown that it is Zariski closed. Show that there is no decomposition $X = F_1 \cup F_2$ of X as the union of two Zariski closed sets. One says that X is then irreducible. *(Begin by showing that the set of matrices having T^n as minimal polynomial is dense in X. Use Jordan's normal form, study the space* $\mathrm{GL}_n(k)$*, and use things)*