ENS de Lyon
M1

Géométrie Algébrique
2025-2026

TD2 : Conics and more commutative algebra

Exercises with a  are algebraic geometry exercices which will be corrected during the exercise session, if time allows it. Exercises with a  are important exercices of commutative algebra.

Exercise 1. (*Conics*) —

1.

a. Solve $x^2 + y^2 = 3$ in $\mathbb{Q}^2$.

b. Solve $x^2 + y^2 = 5$ in $\mathbb{Q}^2$.

c. For which primes $p \in \mathbb{Z}_{>0}$ does the equation $x^2 + y^2 = p$ have any solution ? Infinitely many solutions ?

2. Prove the classification of the conics. Namely, let $ax^2+bxy+cy^2+dx+ey+f = 0$ ($a, b, c, d, e, f \in \mathbb{R}$) be the equation of a conic. Prove that, under an affine coordinate change, this conic is an ellipse/parabola/hyperbola (non-degenerate case) or some finite union of affine subspaces (degenerate case).

3. Let k be a field of characteristic $\neq 2$ and V a 3-dimensional k-vector space ; let $Q : V \to k$ be a non-degenerate quadratic form on V.

a. Show that if $0 \neq e_1 \in V$ satisfies $Q(e_1) = 0$, then there exists some basis e_1, e_2, e_3 of V such that $Q(x_1e_1 + x_2e_2 + x_3e_3) = x_1x_3 + ax_2^2$.

b. Prove that a non-empty and non-degenerate conic $C \subset \mathbb{P}^2_k$ is projectively equivalent to $(XZ = Y^2)$.

Exercise 2. (*Ax-Grothendieck theorem*) —

Let $P : k^n \to k^n$ be a polynomial map, where k is an algebraicly closed field. Suppose that P is injective, we want to show that P is surjective also.

1. Show that the fact that P is injective can be expressed by finitely many polynomial relations with coefficients in k.

2. Show that the fact that P misses a point $(a_1, \ldots, a_n) \in k^n$ can be expressed by finitely many polynomial relations with coefficients in k.

3. Denote Λ the image of $\mathbb{Z}$ in k. Show that all the equations of the two previous questions actually have coefficients in a finite type Λ-algebra A (which depends on P and $(a_1, \ldots, a_n)$).

4. Let $\mathfrak{m}$ be a maximal ideal of A. Show that $A/\mathfrak{m}$ is a finite field (you should use a previous result in this sheet).

5. Prove the theorem.

Exercise 3. (*Quadratic nullstellensatz*) — (reference : *Carnets de Voyages en Algébrie*, Philippe Caldero and Marie Peronnier). Let k be of characteristic $\neq 2$, E a finite-dimensional vector space and q, q' two non-degenerate quadratic forms on E. Denote ϕ, ϕ' their polar form. Denote $\mathcal{C}_q = \{u \in E \mid q(u) = 0\}$.

Suppose that there exists some $u \in \mathcal{C}_q$. Suppose that $\mathcal{C}_q = \mathcal{C}_{q'}$, let's show that q and q' are proportional. Denote H_u (resp. H'_u) the hyperplane orthogonal to u for q (resp. q').

1. Let $v \in E$ non zero and $D_{u,v} = \{\alpha u + v, \alpha \in k\}$.

a. Suppose here $v \in H_u$. Show that $\mathcal{C}_q \cap D_{u,v}$ is $D_{u,v}$ if $v \in \mathcal{C}_q$, and $\emptyset$ if not.

b. Suppose $v \notin H_u$. Show that $\mathcal{C}_q \cap D_{u,v}$ is a singleton, compute it.

2. **a.** Deduce that $H_u = H'_u$. Deduce that there exists $\lambda_u \in k^*$ such that $\phi'(u, .) = \lambda_u \phi(u, .)$.

b. Deduce that $\forall v \in E - H_u$, $q'(v) = \lambda_u q(v)$. Conclude when k is $\mathbb{R}$ or $\mathbb{C}$.

3. Show that $\phi'(v, w) = \lambda_u \phi(v, w)$ for all $v, w \notin H_u$. (Show that either $v+w \notin H_u$ or $v + 2w \notin H_u$.)

4. Conclure (build a basis for E that would allow you to use previous questions).

5. Construct a counter-example with $k = \mathbb{R}$ if we relax the hypothesis $\mathcal{C}_q \neq 0$.