TD12 : Essentially, what is a scheme

Exercises with a  are algebraic geometry exercices which will be corrected during the exercise session, if time allows it. Exercises with a  are important exercices of commutative algebra.

Sheaves (faisceaux) are objects that allow to formalize the link between local properties and global properties of a topological space. They allow to give a geometry to a space. So let's give to $\mathrm{Spec}(A)$ a sheaf today.

When we say a sheaf, it is a sheaf of sets or groups or rings.

**Exercise 1.** (*Stalks and germs*) —

Let X be a topological space, $\tau(X)$ the set of its open subsets. Let $\mathcal{F}$ be a presheaf on X. Let $p \in X$. Define the *stalks* of $\mathcal{F}$ at p to be the quotient of the set $\{(f,U) \mid p \in U \in \tau(X), f \in \mathcal{F}(U)\}$ by the following equivalence relation : $(f,U) \sim (g,V)$ iff there is some $p \in W \subset U \cap V$, open, such that $f_{|W} = g_{|W}$.

1. Show that the said equivalence relation used here is indeed an equivalence relation.

2. (properties of the stalk) Show that $\mathcal{F}_p$ is the colimit of the collection of the $\mathcal{F}(U)$, for each open set U containing p. It means the following :
- for every U containing p, we have a morphism $\mathcal{F}(U) \to \mathcal{F}_p$,
- for every $V \subset U$, the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ commutes with those morphisms $\mathcal{F}(U) \to \mathcal{F}_p$ and $\mathcal{F}(V) \to \mathcal{F}_p$,
- (don't do it today) for every other set/ring/group G with such morphisms, those morphisms pass through $\mathcal{F}_p$. It means that $\mathcal{F}_p$ is the "closest possible object to the diagram $(\mathcal{F}(U))_{U \ni p}$".

3. (stalks determine sections) Let U be open in X. Show that the natural map $\mathcal{F}(U) \to \prod_{p \in U} \mathcal{F}_p$ is injective (its image is sometimes called the set of families of compatible germs. They can be defined without knowing that $\mathcal{F}$ is a sheaf, hence are used for sheafification. We will later use a similar process).

4. (morphism induced to stalks) Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves, where $\mathcal{F}, \mathcal{G}$ are sheaves on X. Show that for all $p \in X$, there is a morphism $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ such that $\forall f_p \in \mathcal{F}_p$ (f is an element of some $\mathcal{F}(U)$ where U is an open set containing p), one has

$$\phi_p(f_p) = (\phi(f))_p.$$

Write a commutative diagram to express that relation.

5. Let's show that the stalks keep the information of isomorphisms.

a. Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on X. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves, such that for all open set U of X, $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism. Show that ϕ yields an isomorphism $\phi : \mathcal{F} \simeq \mathcal{G}$.

b. Let $\phi : \mathcal{F} \simeq \mathcal{G}$ be a morphism of sheaves. Suppose that for all $p \in X$, ϕ induces an isomorphism $\mathcal{F}_p \to \mathcal{G}_p$. Show that ϕ is an isomorphism.

Remark for profesionnals : it is still (very easily) true if you replace "isomorphism" by "injective map", but not if you replace by "surjective maps" : you will only get an epimorphism of sheaves, and this is not the same thing as being surjective "for every U".

Exercise 2. (*Sheaves and basis of the topology*) —

Let X be a topological space, and let $\mathcal{B}$ be a *basis* of the topology of X. This means that $\mathcal{B}$ is a subset of the set of open sets of X and that any open subset of X is the union of elements of $\mathcal{B}$. The elements of $\mathcal{B}$ are called *basic* open sets.

1. Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X. Assume for each $U \in \mathcal{B}$, there exists an isomorphism $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for every inclusion $V \subset U$ of elements of $\mathcal{B}$, the following diagram is commutative :

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V). \end{array}$$

Show that $\mathcal{F} \simeq \mathcal{G}$.

Hence the data of a sheaf can be compressed by just keeping the data on a basis.

We define a *presheaf* $\mathcal{F}$ *on the basis* $\mathcal{B}$ to be a contravariant functor from the partially ordered set $\mathcal{B}$ to the category of sets/rings/groups.

We define a *sheaf* $\mathcal{F}$ *on the basis* $\mathcal{B}$ to be a presheaf on the basis $\mathcal{B}$ such that :

- (base identity axiom) If $B = \cup B_i$, with $B, B_i \in \mathcal{B}$, then two elements $f, g \in \mathcal{F}(B)$ that agree on each B_i are equal.
- (base gluability axiom) Let $B = \cup B_i$, with $B, B_i \in \mathcal{B}$, and $f_i \in \mathcal{F}(B_i)$. Suppose that for all i, j, f_i and f_j agree on any basic open set contained in $B_i \cap B_j$. Then there exists $f \in \mathcal{F}(B)$ such that $\forall i, f_{|B_i} = f_i$.

2. Show that if $\mathcal{F}$ is a sheaf on X, then it induces a sheaf on the base $\mathcal{B}$.

3. If $\mathcal{F}$ is a presheaf on X that is a sheaf on the base $\mathcal{B}$, can we say that $\mathcal{F}$ is a sheaf ?

4. Let $\mathcal{F}$ be a sheaf on the base $\mathcal{B}$. Define the *stalk on a base* $\mathcal{F}_p$ to be the same thing as a stalk, but only using the base, so it is the set of *germs on a base* $g = (f, B)$ for $p \in B$ and $f \in \mathcal{F}(B)$, quotiented by the analog equivalence relation (g denotes germs). If $B \in \mathcal{B}$, show that there is a natural injective map $\mathcal{F}(B) \to \prod_{p \in B} \mathcal{F}_p$.

5. Let $(\mathcal{F}(B))_{B \in \mathcal{B}}$ be a sheaf on $\mathcal{B}$. Show that there exists a sheaf $\mathcal{F}$ on X such that that extend the sheaf we started with. Is it unique up to isomorphism ?

Definition.

A *ringed space* $(X, \mathscr{O}_X)$ is the data of X, a topological space, and of $\mathscr{O}_X$, a sheaf of commutative rings on X. It is said to be a *locally ringed space* if the stalks are local rings.

Exercise 3. (*Affine algebraic varieties are schemes*) — Let k be an algebraicly closed field. Let X be an affine algebraic variety over k. The goal of this exercise is to show that X is naturally a locally ringed space.

1. Let $f \in k[X]$, define $\mathscr{O}_X(D(f)) = k[X]_f$, the localization of $k[X]$ at at the multiplicative subset $\{1, f, f^2, \dots\}$. Show that $\mathscr{O}_X(D(f))$ is well-defined.

2. Show that inclusions $D(f) \subset D(g)$ induce a morphism of rings $\mathscr{O}_X(D(g)) \to \mathscr{O}_X(D(f))$.

3. What is $\mathscr{O}_X(D(f) \cap D(g))$?

4. Show that this defines a sheaf on the basis of the topology given by standard affine subsets $(D(f))_f$ and hence gives a sheaf $\mathscr{O}_X$ on X.

5. If $X \to Y$ is a morphism, what do we get on sheaves ?

6. Let A be any commutative ring. Use the same construction to show that $\mathrm{Spec}(A)$ is naturally a locally ringed space. $\mathscr{O}_{\mathrm{Spec}(A)}$ is called the structural sheaf on A. If $A \to B$ is a ring morphisms, what do we get on spectra and on sheaves ?

This bonus exercise is an example of study of module over a sheaf of rings. Modules over sheaves of rings are important techniques used to study the sheaves of rings themselves.

Definition.

Let $\mathcal{F}$ be a sheaf of abelian groups on the space X. $\mathcal{F}$ is said to be a sheaf of $\mathscr{O}_X$-modules if for every open subset U of X, $\mathcal{F}(U)$ is an $\mathscr{O}_X(U)$-module, and for every inclusion of open subsets $V \subset U$, the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the morphism of rings $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$. A morphism of sheaves of modules on X is a morphism of Abelian sheaves which is a morphism a $\mathscr{O}_X(U)$-modules over each open subset $U \subset X$.

Exercise 4. (*Sheaves of modules on an affine variety*) —

Let k be algebraicly closed. Let X be an affine algebraic variety. Let M be a $k[X]$-module. Let U be an open subset of X.

1. Recall why $(D(f))_{f \in k[X]}$ is a basis of the topology on X.

For each $f \in k[X]$, we define $\widetilde{M}(D(f)) = M_f$, where M_f is the localization of M at at the multiplicative subset $\{1, f, f^2, \dots\}$.

2. Show that $\widetilde{M}$ is well defined. Show that inclusions $D(f) \subset D(g)$ induce a morphism of groups $\widetilde{M}(D(g)) \to \widetilde{M}(D(f))$ compatible with $k[X]_{(g)} \to k[X]_{(f)}$.

3. Show that this defines a sheaf on the basis of the topology given by standard affine subset $(D(f))_f$ and hence gives a sheaf $\widetilde{M}$ on X.

4. Show that $\widetilde{M}$ is a sheaf of $\mathscr{O}_X$-modules.

5. Show that there is a bijection between morphisms of $\mathscr{O}_X$-modules $\widetilde{M} \to \widetilde{N}$ and morphisms of $k[X]$-modules $M \to N$.