ENS de Lyon — Géométrie Algébrique
M1 — 2025-2026

TD12 : Essentially, what is a scheme

Exercises with a  are algebraic geometry exercices which will be corrected during the exercise session, if time allows it. Exercises with a  are important exercices of commutative algebra.

Sheaves (faisceaux) are objects that allow to formalize the link between local properties and global properties of a topological space. They allow to give a geometry to a space. So let's give to Spec(A) a sheaf today.

When we say a sheaf, it is a sheaf of sets (but the same proofs will work for sheaves of groups or sheaves of rings).

Exercise 1. (*Stalks and germs*) —

Let X be a topological space, $\tau(X)$ the set of its open subsets. Let $\mathcal{F}$ be a presheaf on X. Let $p \in X$. Define the *stalks* of $\mathcal{F}$ at p to be the quotient of the set $\{(f, U) \mid p \in U \in \tau(X), f \in \mathcal{F}(U)\}$ by the following equivalence relation : $(f, U) \sim (g, V)$ iff there is some $p \in W \subset U \cap V$, open, such that $f_{|W} = g_{|W}$.

1. Show that the said equivalence relation used here is indeed an equivalence relation.

2. (properties of the stalk) Show that $\mathcal{F}_p$ is the colimit of the collection of the $\mathcal{F}(U)$, for each open set U containing p. It means the following :

— for every U containing p, we have a morphism $\mathcal{F}(U) \to \mathcal{F}_p$,

— for every $V \subset U$, the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ commutes with those morphisms $\mathcal{F}(U) \to \mathcal{F}_p$ and $\mathcal{F}(V) \to \mathcal{F}_p$,

— (don't do it today) for every other set/ring/group G with such morphisms, those morphisms pass through $\mathcal{F}_p$. It means that $\mathcal{F}_p$ is the "closest possible object to the diagram $(\mathcal{F}(U))_{U \ni p}$".

3. (stalks determine sections) Let U be open in X. Show that the natural map $\mathcal{F}(U) \to \prod_{p \in U} \mathcal{F}_p$ is injective (its image is sometimes called the set of families of compatible germs. They can be defined without knowing that $\mathcal{F}$ is a sheaf, hence are used for sheafification. We will later use a similar process).

4. (morphism induced to stalks) Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves, where $\mathcal{F}, \mathcal{G}$ are sheaves on X. Show that for all $p \in X$, there is a morphism $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ such that $\forall f_p \in \mathcal{F}_p$ (f is an element of some $\mathcal{F}(U)$ where U is an open set containing p), one has

$$\phi_p(f_p) = (\phi(f))_p.$$

Write a commutative diagram to express that relation.

Correction. Soit $\phi : \mathcal{F} \to \mathcal{G}$ un morphisme de faisceaux sur X.

Soit $f \in \mathcal{F}_p$. On peut définir $\phi(f)$ de la façon suivante : on prend V où f admet un représentant, et on prend $\phi_V(f)$. C'est un élément de $\mathcal{G}(V)$ donc on peut prendre son germe en p.

Montrons que ça ne dépend pas du choix du représentant. Si $f = f'$ en tant que germes en p, où f est définie sur V et f' sur V', alors on doit avoir un voisinage de p W tq $f_{|W} = f'_{|W}$. Le germe en p de $\phi_V(f)$ est clairement égal à celui de $\phi_W(f)$ donc les germes de $\phi_V(f)$ et $\phi_{V'}(f')$ sont égaux.

On obtient bien une application $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ telle que $\forall f \in \mathcal{F}(V)$ où $V \ni p$, $\phi_p(f_p) = \phi_V(f)_p$. De façon grossière : $\boxed{\phi(f_p) = \phi(f)_p}$. C'est bien le fait que le diagramme commute.

5. Let's show that the stalks keep the information of isomorphisms.

a. Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on X. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves, such that for all open set U of X, $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism. Show that ϕ yields an isomorphism $\phi : \mathcal{F} \simeq \mathcal{G}$.

Correction. On appelle ψ_U les inverses de f_U. On va définir un morphisme de préfaisceaux $\mathcal{G} \to \mathcal{F}$. Si U est ouvert, alors on lui associe l'application $\phi_U : \mathcal{G}(U) \to \mathcal{F}(U)$. Montrons que c'est bien une transformation naturelle de foncteurs. Si $U \subset U'$,

$$\begin{array}{ccc} \mathcal{G}(U') & \xrightarrow[res]{} & \mathcal{G}(U) \\ \uparrow{\scriptstyle \phi_{U'}} & & {\scriptstyle \phi_U}\uparrow \\ \mathcal{F}(U)' & \xrightarrow{res} & \mathcal{F}(U) \end{array} \qquad \begin{array}{ccccccc} \mathcal{G}(U') & \xrightarrow{id} & \mathcal{G}(U') & \xrightarrow{res} & \mathcal{G}(U) & & \\ & {\scriptstyle \psi_{U'}}\searrow & \uparrow{\scriptstyle \phi_{U'}} & & {\scriptstyle \phi_U}\uparrow & \searrow{\scriptstyle \psi_U} & \\ & & \mathcal{F}(U)' & \xrightarrow[res]{} & \mathcal{F}(U) & \xrightarrow{id} & \mathcal{G}(U) \end{array}$$

Le premier diagramme commute par hypothèse. Donc le deuxième aussi. Et c'est ce qu'on voulait.

Maintenant, il est clair que la composition de ϕ et ψ fait l'identité dans les deux sens.

b. Let $\phi : \mathcal{F} \simeq \mathcal{G}$ be a morphism of sheaves. Suppose that for all $p \in X$, ϕ induces an isomorphism $\mathcal{F}_p \to \mathcal{G}_p$. Show that ϕ is an isomorphism.

Remark for profesionnals : it is still (very easily) true if you replace "isomorphism" by "injective map", but not if you replace by "surjective maps" : you will only get an epimorphism of sheaves, and this is not the same thing as being surjective "for every U".

Correction. On va prouver que les ϕ_U sont injectives. On regarde juste :

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \prod_{p \in U} \mathcal{F}_p & \hookrightarrow & \prod_{p \in U} \mathcal{G}_p \end{array}$$

et le tour est joué.

On peut prouver que les ϕ_U sont surjectives. Soit $g \in \mathcal{G}(U)$. On obtient alors une famille $(g_p)_{p \in U}$ qui admet un antécédent par les ϕ_p : on note ces germes antécédents $(f^{(p)})_p$ dont on choisit des représentants $f^{(p)}$ qui sont des sections définies sur un ouvert $p \in U_p \subset U$. Le fait que $\phi(f_p^{(p)}) = g_p$ implique l'égalité $\phi(f^{(p)})_p = g_p$ en tant que germes. Cela signifie qu'il existe V_p contenant p tel que $\phi(f^{(p)})_{|V_p}) = g_{|V_p}$. On remarque que les V_p recouvrent U. Posons $f'^{(p)} = f^{(p)}_{|V_p}$, de telle façon que $\phi_{V_p}(f'^{(p)}) = g_{|V_p}$.

Montrons que les $f'^{(p)}$ sont compatibles, i.e., si $p, p' \in U$, montrons que $f'^{(p)}_{|V_p \cap V_{p'}} = f'^{(p')}_{|V_p \cap V_{p'}}$.

Il suffit pour cela de remarquer que

$$\begin{aligned} \phi_{V_p \cap V_{p'}}(f'^{(p)}_{|V_p \cap V_{p'}}) &= \phi_{V_p}(f'^{(p)})_{|V_p \cap V_{p'}} \\ &= g_{|V_p \cap V_{p'}}. \end{aligned}$$

La même chose s'applique à $\phi_{V_p \cap V_{p'}}(f'^{(p')}_{|V_p \cap V_{p'}})$ donc $\phi_{V_p \cap V_{p'}}(f'^{(p)}_{|V_p \cap V_{p'}}) = \phi_{V_p \cap V_{p'}}(f'^{(p')}_{|V_p \cap V_{p'}})$. Par injectivité de $\phi_{V_p \cap V_{p'}}$, on obtient l'égalité voulue. Il existe donc un recollement $f \in \mathcal{F}(U)$.

Montrons que $\phi_U(f) = g$. Il suffit de le montrer sur les V_p car c'est un recouvrement. On a $\phi_U(f)_{|V_p} = \phi_{V_p}(f_{V_p}) = \phi_{V_p}(f'^{(p)}) = g_{V_p}$.


Exercise 2. (*Sheaves and basis of the topology*) —

Let X be a topological space, and let $\mathcal{B}$ be a *basis* of the topology of X. This means that $\mathcal{B}$ is a subset of the set of open sets of X and that any open subset of X is the union of elements of $\mathcal{B}$. The elements of $\mathcal{B}$ are called *basic* open sets.

1. Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X. Assume for each $U \in \mathcal{B}$, there exists an isomorphism $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for every inclusion $V \subset U$ of elements of $\mathcal{B}$, the following diagram is commutative :

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V). \end{array}$$

Show that $\mathcal{F} \simeq \mathcal{G}$.

Correction. ▷ Let U be an open set of X, we want to define an isomorphism $f : \mathcal{F}(U) \to \mathcal{G}(U)$. For now, let's just define a morphism (we will just use the fact that f is a morphism of sheaves). Consider the family $(B_i)_i$ of all the basic open sets included in U, so that $U = \bigcup_i B_i$. If $s \in \mathcal{F}(U)$, one has, for each i, $s_{|B_i} \in \mathcal{F}(B_i)$, hence one has elements $f_{B_i}(s_{|B_i})$ of $\mathcal{G}(B_i)$. Denote, for simplicity, $s_{|B_i} =: s_i$, and $s'_i = f_{B_i}(s_i)$.

In order to create an element of $\mathcal{G}(U)$, it suffices to show that $\forall i, j$, $s'_{i|B_j \cap B_i} = s'_{j\,B_j \cap B_i}$, which implies that one can glue the s'_i together. The problem here is that $B_i \cap B_j$ is not necessarily in $\mathcal{B}$. Let i, j, fixed. Let $V \in \mathcal{B}$ contained in $B_i \cap B_j$. Let's show that $s'_{i,|V} = s'_{j,|V}$.

By the diagram, applied to the open sets $V \subset B_i$, and to the section s_i, one has $f_{B_i}(s_i)_{|V} = f_V(s_{i|V})$. Hence $s'_{i,V} = f_V(s_{i|V})$. This is also true for j. Furthermore, $s_{i|V} = s_{j|V}$ because $\mathcal{F}$ is a (pre-)sheaf. Hence $s'_{i,V} = f_V(s_{i|V}) = f_V(s_{j|V}) = s'_{j,V}$. This is what we wanted, and, as the V cover $B_i \cap B_j$, this implies that $s'_{i,B_j \cap B_i} = s'_{j,B_j \cap B_i}$.

Hence, as $\mathcal{G}$ is a sheaf, the s'_i can be glued together to create an element $s' \in \mathcal{G}(U)$, such that $\forall i, s'_{|B_i} = s'_i = f_{B_i}(s_{|B_i})$. Denote $f(s) = s'$.

Hence the map $f : \mathcal{F}(U) \to \mathcal{G}(U)$ is well-defined (because we did not make any choice), we will denote it f_U. It is easy to check that if U is a basic open set, then the newly constructed f_U is equal to the original f_U, so there is no ambiguity. We write here that we know that $\forall i, (f_U(s))_{|B_i} = f_{B_i}(s_{|B_i})$.

▷ One should take time to check that f_U is indeed a set/group/ring morphism. For sets, nothing to do, but for group/ring property, take the time to do it using the *sheaf identity axiom* (and the fact that you know that the f_B, for each $B \in \mathcal{B}$, are already group/ring morphisms).

▷ We still need to show that f is a morphism of sheaves ! Let $V \subset U$ be any open sets, let's show that f commutes with the restrictions. We just need to show that $(f_U(s))_{|V} = f_V(s_{|V})$.

By definition of $f_U(s)$, we know that for all basic open set $W \subset U$, $(f_U(s))_{|W} = f_W(s_{|W})$. In the same manner, by definition of $f_V(s_{|V})$, for all basic open set $W \subset V$, $(f_V(s_{|V}))_{|W} = f_W((s_{|V})_{|W})$, which is just equal to $f_W(s_{|W})$.

Hence, for all basic open subset $W \subset V$, we have $(f_V(s_{|V}))_{|W} = f_W(s_{|W}) = (f_U(s))_{|W} = \left((f_U(s))_{|V}\right)_{|W}$. By the sheaf identity axiom on $\mathcal{G}$, this means that $f_V(s_{|V}) = (f_U(s))_{|V}$, this is what we wanted.

▷ We still need to show that $f : \mathcal{F} \to \mathcal{G}$ is an isomorphism. The trick is to apply the previous result to f^{-1}. The hypothesis gives us morphisms $(f_U)^{-1} : \mathcal{G}(U) \to \mathcal{F}(U)$ such that, for every inclusion $V \subset U$, we have the same diagram (but swapped left to right) that commutes. So we just proved that there is a morphism of sheaves, call it $g : \mathcal{G} \to \mathcal{F}$, such that, for every $B \in \mathcal{B}$, $g_B = (f_B)^{-1}$.

Let U be an open set of X. We just need to show that $g_U \circ f_U = \mathrm{id}_{\mathcal{F}(U)}$ and that $f_U \circ g_U = \mathrm{id}_{\mathcal{G}(U)}$. Let $s \in \mathcal{F}(U)$. For every W basic contained in U, one has $g_W \circ f_W(s_{|W}) = s_{|W}$. As g and f are morphisms of sheaves, they commute with the restriction so we just have $(g_U \circ f_U(s))_{|W} = s_{|W}$, hence equality. The same thing works for the other way round.

The conclusion is that g is the inverse sheaf morphism to f hence the result. □

Hence the data of a sheaf can be compressed by just keeping the data on a basis.

We define a *presheaf $\mathcal{F}$ on the basis $\mathcal{B}$* to be a contravariant functor from the partially ordered set $\mathcal{B}$ to the category of sets/rings/groups.

We define a *sheaf $\mathcal{F}$ on the basis $\mathcal{B}$* to be a presheaf on the basis $\mathcal{B}$ such that :

- (base identity axiom) If $B = \cup B_i$, with $B, B_i \in \mathcal{B}$, then two elements $f, g \in \mathcal{F}(B)$ that agree on each B_i are equal.
- (base gluability axiom) Let $B = \cup B_i$, with $B, B_i \in \mathcal{B}$, and $f_i \in \mathcal{F}(B_i)$. Suppose that for all i, j, f_i and f_j agree on any basic open set contained in $B_i \cap B_j$. Then there exists $f \in \mathcal{F}(B)$ such that $\forall i, f_{|B_i} = f_i$.

2. Show that if $\mathcal{F}$ is a sheaf on X, then it induces a sheaf on the base $\mathcal{B}$.

Correction. I think it is easy.

3. If $\mathcal{F}$ is a presheaf on X that is a sheaf on the base $\mathcal{B}$, can we say that $\mathcal{F}$ is a sheaf ?

Correction. The base identity/gluability axioms are not strong enough to tackle general open sets : it only works for coverings of basic open sets ! Here is an example : $X = \{0, 1\}$ with the discrete topology. Then $\mathcal{B} = \{\{0\}, \{1\}\}$ is a basis for X. Consider the following presheaf $\mathcal{F}$ on X : let f and g be two different objects, define $\mathcal{F}(X) = \{f, g\}$, $\mathcal{F}(\{0\}) = \mathcal{F}(\{1\}) = \mathcal{F}(\emptyset) = \{f\}$ (with the obvious restriction maps). It is clearly not a sheaf because the sections on X, f and g, agree on $\{0\}$ and on $\{1\}$ but are not equal. But $\mathcal{F}$ is a sheaf on the base $\mathcal{B}$ because the quantification "for all coverings $B = \cup B_i$, with $B, B_i \in \mathcal{B}$" only applies to trivial coverings.

4. Let $\mathcal{F}$ be a sheaf on the base $\mathcal{B}$. Define the *stalk on a base* $\mathcal{F}_p$ to be the same thing as a stalk, but only using the base, so it is the set of *germs on a base* $g = (f, B)$ for $p \in B$ and $f \in \mathcal{F}(B)$, quotiented by the analog equivalence relation (g denotes germs). If $B \in \mathcal{B}$, show that there is a natural injective map $\mathcal{F}(B) \to \prod_{p \in B} \mathcal{F}_p$.

5. Let $(\mathcal{F}(B))_{B \in \mathcal{B}}$ be a sheaf on $\mathcal{B}$. Show that there exists a sheaf $\mathcal{F}$ on X such that that extend the sheaf we started with. Is it unique up to isomorphism ?

Correction. ▷ Unicity up to isomorphism is trivial, it is just a previous question in this exercise. Now, let U be an open set of X, let's define $\mathcal{G}(U)$ (we will denote our sheaf $\mathcal{G}$ to make things clearer).

The following process looks like the sheafification process : in order to create a sheaf out of things that are not a sheaf, take families of germs that, at least locally, make sense (an element of a sheaf). If $(g_p \in \mathcal{F}_p)_{p \in U}$ is such a family of germs, one says it is a family of *compatible* germs if for all $p \in U$, there is a neighbourhood $p \in B \in \mathcal{B}$, $B \subset U$ on which we can find a section $s \in \mathcal{F}(B)$ such that $\forall q \in B$, $s_q = g_q$, as germs at q (it means that our family of germs look like s). Denote $\mathcal{G}(U)$ the set of all families of compatible germs.

▷ Let's prove, firstly, for sanity check, that for all $B \in \mathcal{B}$, the natural map $\mathcal{F}(B) \to \mathcal{G}(B)$ is an isomorphism. It is clearly injective because $\mathcal{G}(B)$, by definition, injects into $\prod_{p \in B} \mathcal{F}_p$. The composition $\mathcal{F}(B) \to \mathcal{G}(B) \to \prod_{p \in B} \mathcal{F}_p$ is the natural map $\mathcal{F}(B) \to \prod_{p \in B} \mathcal{F}_p$, which is injective because of the previous question.

It is surjective because : if $(g_p)_{p \in B}$ is a family of compatible germs, then at every point p, there is some basic open set V and $s \in \mathcal{F}(V)$ such that $\forall q \in V$, $s_q = g_q$, as germs at q. Those basic open sets V cover B. Hence, we have a partition of B by open sets that we denote $(B_i)_i$, and a family of sections $(s^{(i)} \in \mathcal{F}(B_i))_i$, such that $\forall q \in B_i$, $s^{(i)}_q = g_q$.

It is natural to try to glue the $s^{(i)}$ together. Let i, j two indices, let's show that "$s^{(i)}_{|B_i \cap B_j} = s^{(j)}_{|B_i \cap B_j}$" (this does not make any sense but this is exactly the idea you want to use here. $B_i \cap B_j$ is not necessarily in $\mathcal{B}$ so reduce to a partition of $B_i \cap B_j$ using elements of $\mathcal{B}$. The argument will work). But we know that $\forall q \in B_i \cap B_j$, we have $s^{(i)}_q = g_q = s^{(j)}_q$. Hence $s^{(i)}_{|B_i \cap B_j}$ and $s^{(j)}_{|B_i \cap B_j}$ agree on the stalks hence are equal ($\mathcal{F}$ is a sheaf on a basis). Hence there is $s \in \mathcal{F}(B)$ such that $\forall i, s_{|B_i} = s^{(i)}$. This implies that for every $p \in B$, p being in some B_i, for some i, one has $s_p = s^{(i)}_p = g_p$. This means that s is sent to $(g_p)_p$ by the map $\mathcal{F}(B) \to \mathcal{G}(B)$ is surjective.

▷ Let's prove that it is a sheaf. Later.

Definition.

A *ringed space* $(X, \mathscr{O}_X)$ is the data of X, a topological space, and of $\mathscr{O}_X$, a sheaf of commutative rings on X. It is said to be a *locally ringed space* if the stalks are local rings.

Exercise 3. (*Affine algebraic varieties are schemes*) — Let k be an algebraicly closed field. Let X be an affine algebraic variety over k. The goal of this exercise is to show that X is naturally a locally ringed space.

1. Let $f \in k[X]$, define $\mathscr{O}_X(D(f)) = k[X]_f$, the localization of $k[X]$ at at the multiplicative subset $\{1, f, f^2, \dots\}$. Show that $\mathscr{O}_X(D(f))$ is well-defined.

Correction. By nullstellensatz, if f and g conflict, then some n makes $f^n \in (g)$ hence $\{1, f, f^2, \dots\}$ contains a multiple of g so g is invertible in $k[X]_f$, and conversely. More precisely, $k[X]_f$ can be seen as the localization of $k[X]$ at the set of functions that don't vanish on $V(f)$. This is well-defined!

2. Show that inclusions $D(f) \subset D(g)$ induce a morphism of rings $\mathscr{O}_X(D(g)) \to \mathscr{O}_X(D(f))$.

3. What is $\mathscr{O}_X(D(f) \cap D(g))$?

4. Show that this defines a sheaf on the basis of the topology given by standard affine subsets $(D(f))_f$ and hence gives a sheaf $\mathscr{O}_X$ on X.

Correction. One must show the axioms for a sheaf on a basis. For the gluability, take advantage of the fact that the definition of the gluability axiom is simplified, because the basis is stable by finite intersection. See p133 in Vakil.

5. If $X \to Y$ is a morphism, what do we get on sheaves?

6. Let A be any commutative ring. Use the same construction to show that $\mathrm{Spec}(A)$ is naturally a locally ringed space. $\mathscr{O}_{\mathrm{Spec}(A)}$ is called the structural sheaf on A. If $A \to B$ is a ring morphisms, what do we get on spectra and on sheaves?

This bonus exercise is an example of study of module over a sheaf of rings. Modules over sheaves of rings are important techniques used to study the sheaves of rings themselves.

Definition.

Let $\mathcal{F}$ be a sheaf of abelian groups on the space X. $\mathcal{F}$ is said to be a sheaf of $\mathscr{O}_X$-modules if for every open subset U of X, $\mathcal{F}(U)$ is an $\mathscr{O}_X(U)$-module, and for every inclusion of open subsets $V \subset U$, the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the morphism of rings $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$. A morphism of sheaves of modules on X is a morphism of Abelian sheaves which is a morphism a $\mathscr{O}_X(U)$-modules over each open subset $U \subset X$.

Exercise 4. (*Sheaves of modules on an affine variety*) —

Let k be algebraicly closed. Let X be an affine algebraic variety. Let M be a $k[X]$-module. Let U be an open subset of X.

1. Recall why $(D(f))_{f \in k[X]}$ is a basis of the topology on X.

For each $f \in k[X]$, we define $\widetilde{M}(D(f)) = M_f$, where M_f is the localization of M at at the multiplicative subset $\{1, f, f^2, \dots\}$.

2. Show that $\widetilde{M}$ is well defined. Show that inclusions $D(f) \subset D(g)$ induce a morphism of groups $\widetilde{M}(D(g)) \to \widetilde{M}(D(f))$ compatible with $k[X]_{(g)} \to k[X]_{(f)}$.

3. Show that this defines a sheaf on the basis of the topology given by standard affine subset $(D(f))_f$ and hence gives a sheaf $\widetilde{M}$ on X.

4. Show that $\widetilde{M}$ is a sheaf of $\mathscr{O}_X$-modules.

5. Show that there is a bijection between morphisms of $\mathscr{O}_X$-modules $\widetilde{M} \to \widetilde{N}$ and morphisms of $k[X]$-modules $M \to N$.