ENS de Lyon
M1

Géométrie Algébrique
2025-2026

TD11 : Dimension theory

Exercises with a  are algebraic geometry exercices which will be corrected during the exercise session, if time allows it. Exercises with a  are important exercices of commutative algebra.

**Exercise 1.** (*Dimension but without dimension*) — If R is any ring, we denote $R[[t]]$ the ring of formal power series $\{\sum_i a_i t^i \mid a_i \in R\}$.

Let A be a graded k-algebra, $A = \bigoplus_{D \geq 0} A_D$. We define the associated Hilbert series $H(A,t) = \sum_{D=0}^{\infty} \dim_k(A_D) t^D$ as a formal series in t with coefficients in $\mathbb{R}$ (it is well-defined, assuming that the $\dim_k(A_D)$ are finite).

If f and g are two real formal series, we denote that $f \leq g$ when the coefficients of g are greater or equal to those of f, after a certain rank.

1. If $A = k[X_0, \ldots, X_n]$, compute $H(A,t)$.

2. If A is a graded k-algebra with $H(A,t)$ being well-defined. Let $x \in A$ an homogeneous element of degree d. Show that $H(A/x,t) \geq (1-t^d)H(A,t)$.

3. Let $P_1, \ldots, P_m$ be homogeneous polynomials in $A := k[X_0, \ldots, X_n]$, of respective degrees $d_1, \ldots, d_m$. Denote $B = A/(P_1, \ldots, P_m)$. Show that $H(B,t) \geq \frac{\prod_{i=1}^m (1-t^{d_i})}{(1-t)^n}$.

4. Deduce that B is a k-vector space of infinite dimension whenever $m < n+1$.

5. Suppose that k is algebraicly closed. Deduce that $V_+(P_1, \ldots, P_m)$ is not empty.

Définition. Let A be a commutative ring. The Krull dimension of A is the supremum of the length of all chains of proper prime ideals. (we say that a chain with only 1 element is of length 0)

**Exercise 2.** (*Anneaux de dimension 0*) — Soit A un anneau noetherien de dimension 0. On veut montrer que A est *artinien*, c'est à dire que toute suite décroissante d'idéaux de A stationne.

On pourra utiliser sans preuve le fait qu'un anneau noetherien possède un nombre fini d'idéaux premiers minimaux.

1. Montrez que A n'a qu'un nombre fini d'idéaux maximaux (qui sont aussi premiers minimaux).

2. On note $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ les ideaux maximaux de A. Montrez qu'il existe un entier $k \in \mathbb{N}^*$ tel que A soit isomorphe au produit $\prod_i A/{\mathfrak{m}_i}^k$. Et que chacun des $A/{\mathfrak{m}_i}^k$ est local.

3. Soit R un anneau local noetherien tel dont l'idéal maximal est nilpotent. Montrez que R est Artinien. (utiliser la longueur d'un module : si M est un R-module sur un anneau commutatif R, on définit $l(M)$ comme étant le supremum des longueurs de chaînes de sous-modules dans M. Si $0 \to M' \to M \to M'' \to 0$ est une suite exacte, montrer que $l(M) = l(M') + l(M''))$

4. En déduire que A est Artinien.

5. Donner un fermé de $\mathbb{A}_k^n$ dont l'anneau des fonctions est de dimension 0.

Exercise 3. (*Krull's Hauptidealsatz*) — Let A be a Noetherian ring and let $\mathfrak{p}$ be a prime ideal of A. The height of $\mathfrak{p}$, denoted by $\mathrm{ht}(\mathfrak{p})$ is the maximal lenght of a decreasing chain of prime ideals $\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_r$.

The goal of this exercise is to show the following fundamental theorem : If $f \in A \setminus (A^\times \cup \{0\})$, then any prime ideal $\mathfrak{p}$ of A containing f and minimal for this property has height 1.

1. We take f and $\mathfrak{p}$ as in the statement.

a. Show that to prove the statement, it is enough to prove it when A is local with maximal ideal $\mathfrak{p}$. In that setting, show that $\dim A/(f) = 0$.

b. Let $\mathfrak{q}$ a prime ideal of A stricly contained in $\mathfrak{p}$. By assumption $f \notin \mathfrak{q}$. What do we have to show on $\mathfrak{q}$?

c. Consider the decreasing sequence of ideals $\mathfrak{q}_n := \mathfrak{q}^n A_\mathfrak{q} \cap A = \{a \in A \mid \exists s \in A \setminus \mathfrak{q}, sa \in \mathfrak{q}^n A_\mathfrak{q}\}$. Show that $(\mathfrak{q}_n \mod (f))$ is stationnary, and then show that $(\mathfrak{q}_n)$ is stationary.

d. Using Nakayama's lemma in $A_\mathfrak{q}$, show that $\mathfrak{q} = 0$.

2. Let A be a local and Notherian ring. Let $I = (f_1, \ldots, f_n)$ an ideal of A. Show that $\dim A \leqslant \dim A/I + r$. Deduce that the dimension of A is less than the dimension as a $k = A/\mathfrak{m}$ vector space of $\mathfrak{m}/\mathfrak{m}^2$. In particular, the dimension of A is finite.

Exercise 4. (*An application of the Hauptidealsatz*) —

Let X be an affine irreducible variety. Let $f \in \mathscr{O}_X(X)$ be a regular function on X (just think of a polynomial function on X) which is not invertible and not zero. Show that every irreducible component of $V(f) \subset X$ has dimension $\dim(X) - 1$.

Exercise 5. (*Hypersurfaces, codimension* 1 *and a trap.*) —

1. Let X be an affine algebraic variety such that $\mathscr{O}_X(X)$ is a factorial ring.

a. Let $f \in \mathscr{O}_X(X)$ irreducible. Show that $\dim V(f) = \dim X - 1$ (for this you do not need the Hauptidealsatz.)

b. Let $Y \subset X$ an irreducible closed subset of X such that $\dim Y = \dim X - 1$. Show that there exists $f \in \mathscr{O}_X(X)$ irreducible such that $Y = V(f)$.

2. Let $X = V(ad - bc) \subset \mathbb{A}_k^4$ and let $Y = V(a, b) \subset X$. Show that Y is a closed irreducible closed subset of X such that $\dim Y = \dim X - 1$. Show that there are no $f \in \mathscr{O}_X(X)$ such that $Y = V(f)$.