TD11 : BIRATIONAL MAPS, TANGENT SPACES, DIMENSION

Définition. Un *idéal premier minimal* d'un anneau A est un élément minimal pour l'inclusion de $\mathrm{Spec}(A)$ (l'ensemble des idéaux premiers de A).

Le spectre $\mathrm{Spec}(A)$ est muni de la topologie de Zariski : les fermés sont les ensembles, pour chaque idéal I de A, $V(I) = \{\mathfrak{p} \in \mathrm{Spec}\, A \mid f \in I\}$.

On vérifie aisément que $V(I) \cap V(J) = V(I+J)$ et $V(I) \cup V(J) = V(I \cap J)$.

L'exercise suivant n'a pas été corrigé en classe mais peut aider à comprendre des choses, pour vos révisions. On va employer ses résultats dans la suite.

Exercise 1. (*Topologie du spectre d'un anneau noethérien*) — **1.** Soit A un anneau noethérien. Montrer que l'espace topologique $\mathrm{Spec}\, A$ est noetherien, au sens suivant : toute chaîne descendante de fermés stationne.

Correction. C'est beaucoup plus facile que ce que vous croyez ! Les fermés de $\mathrm{Spec}\, A$ sont en bijection décroissante avec les idéaux de I !

2. Montrez que toute partie d'un espace topologique noethérien est un espace noethérien.

Correction. Définition de la topologie induite.

3. Soit A un anneau quelconque. Si I est un idéal de A montrer que $\mathrm{Spec}(A/I)$ est homéomorphe à $V(I)$ dans $\mathrm{Spec}\, A$.

Correction. Si $\mathfrak{p} \in \mathrm{Spec}(A/I)$, alors en notant $\pi : A \to A/I$ la projection, on a $\pi^{-1}(\mathfrak{p}) \in V(I)$. De là, ce n'est pas trop difficile, vous pouvez passer cette question pour gagner du temps dans vos révisions.

4. Let X be noetherian and $\mathcal{A}$ a non-empty set of closed subsets of X. Show that $\mathcal{A}$ has a minimal element.

Correction. Suppose that it's false. Then no element of $\mathcal{A}$ is minimal. Hence, every element of $\mathcal{A}$ contains a strictly smaller element of $\mathcal{A}$. Hence, taking any element $P_0 \in \mathcal{A}$, there is some $P_1 \subsetneq P_0$, $P_1 \in \mathcal{A}$. Then one builds by recursion an infinite decreasing sequence $(P_n)_n$ in $\mathcal{A}$, impossible.

5. Show that any noetherian space X has a decomposition $X = \bigcup_i X_i$ where the X_i are irreducible and $\forall i \neq j$, $X_i \not\subseteq X_j$.

Correction. Suppose that the set $\mathcal{A}$ of closed subsets of X that don't have such a decomposition is not empty. Take Y, its minimal element. Then Y is not irreducible hence can we written as $Y_1 \cup Y_2$. It is not hard to see that $Y_1, Y_2 \in \mathcal{A}$. Impossible.

6. Show that the irreducible closed sets in $\mathrm{Spec}(A)$ which are irreducible are precisely those of the form $V(\mathfrak{p})$, where $\mathfrak{p}$ is prime.

Correction. Do it !

7. Deduce that a noetherian ring has a finite number of minimal prime ideals.

8. Show that if A is a noetherian ring, then, for every ideal I of A, the ideal $\sqrt{I}$ decomposes as a finite product of primes, which are minimal primes among the ones containing I.

Correction. Krull gives that $\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}$. But by the previous question applied to Spec(A), one has $V(I) = \bigcup_i V(\mathfrak{p}_i)$, where the $\mathfrak{p}_i$ are primes in A. Hence $\sqrt{I} = \bigcap_i \mathfrak{p}_i$. Let's show now that those $\mathfrak{p}_i$ are minimal among those containing I. We know that $V(\mathfrak{p}_i)$ is a maximal irreducible closed subset of $V(I)$. It just translates to the fact that $\mathfrak{p}_i$ is minimal in Spec(A/I), hence the result.

9. If A is noetherian, and $\mathfrak{p}$ a prime ideal, show that $\mathfrak{p}$ contains one of the minimal primes.

Correction. $\mathfrak{p}$ contains $\sqrt{0}$ hence contains $\bigcap_i \mathfrak{p}_i$, where the $\mathfrak{p}_i$ are the minimal primes of A. Let's deduce that $\mathfrak{p}$ contains one of the $\mathfrak{p}_i$.

In the geometry side, one has $V(\mathfrak{p}) \subset \bigcup_i V(\mathfrak{p}_i)$. We are just reduced to using the unicity of the decomposition in irreducible components. We can juste verify it ad hoc here : the $V(\mathfrak{p}) \cap V(\mathfrak{p}_i)$ are closed and their union is the irreducible set $V(\mathfrak{p})$. Hence one of them has to be $V(\mathfrak{p})$. Hence, for some i, one has $V(\mathfrak{p}) = V(\mathfrak{p}) \cap V(\mathfrak{p}_i)$. But then $V(\mathfrak{p}) \subset V(\mathfrak{p}_i)$, hence $\mathfrak{p}_i \subset \mathfrak{p}$.

10. (assez instructif) Reprouvez les résultats précédents sans utiliser de vocabulaire topologique dans votre démonstration. Pour cela, traduisez chaque étape en propriétés dans des anneaux et des idéaux. Remarquez que l'argument de l'ensemble $\mathcal{A}$ ressemble beaucoup à celui pour l'existence de la décomposition en irréductibles d'un idéal fractionnaire dans un anneau de Dedekind (voir le cours de théorie algébrique des nombres).

Exercise 2. (*Anneaux de dimension 0*) — Soit A un anneau noetherien de dimension 0. On veut montrer que A est *artinien*, c'est à dire que toute suite décroissante d'idéaux de A stationne.

1. Montrez que A n'a qu'un nombre fini d'idéaux maximaux (qui sont aussi premiers minimaux).

Correction. Par hypothèse sur la dimension, les idéaux premiers minimaux sont exactement les idéaux maximaux de A, qui sont aussi simplement exactement les idéaux premiers de A. Or, A possède un nombre fini d'idéaux premiers minimaux.

2. On note $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ les ideaux maximaux de A. Montrez qu'il existe un entier $k \in \mathbb{N}^*$ tel que A soit isomorphe au produit $\prod_i A/{\mathfrak{m}_\mathrm{i}}^k$. Et que chacun des $A/{\mathfrak{m}_\mathrm{i}}^k$ est local.

Correction. Remarquez d'abord que pour tout k, et $i \neq j$, $\mathfrak{m}_i^k + \mathfrak{m}_j^k = A$. En effet, il existe $a \in \mathfrak{m}_i$, $b \in \mathfrak{m}_j$ tels que $a + b = 1$, et mettez ça à la puissance $2k$, d'où la formule. Donc, le théorème des restes chinois nous ramène à montrer qu'il existe un certain k tel que $(\cap_i \mathfrak{m}_i)^k = 0$. Mais par le théorème de Krull, $\cap_i \mathfrak{m}_i = \sqrt{0}$. Or, $\sqrt{0}$ admet un nombre fini de générateurs, $a_1, \dots, a_r$, puisque A est noethérien. Prenez N assez grand tel que $\forall i, a_i^N = 0$. Alors on peut poser $k = Nr + 1$ et vérifier que $\sqrt{0}^k = 0$ (binôme de Newton, ou, pour raccourcir les calculs, vous pouvez utiliser le fait qu'une puissance k-ième d'un polynôme homogène de degré 1 en les a_i est un polynôme homogène de degré k. ça vous ramène à montrer que les monômes de degré $\geq k$ en les a_i sont nuls, et ça, c'est juste de la combinatoire pour utiliser le fait que $a_i^N = 0$).

Les $A/\mathfrak{m}_i^k$ sont locaux d'idéal maximal $\mathfrak{m}_i/\mathfrak{m}_i^k$, ce n'est pas très dur sachant qu'on connaît les idéaux premiers de A et qu'on sait que les $(\mathfrak{m}_j^k)_j$ sont deux à deux comaximaux.

3. Soit R un anneau local noetherien tel dont l'idéal maximal est nilpotent. Montrez que R est Artinien. (utiliser la longueur d'un module : si M est un R-module sur un anneau commutatif R, on définit $l(M)$ comme étant le supremum des longueurs de chaînes de sous-modules dans M. Si $0 \to M' \to M \to M'' \to 0$ est une suite exacte, montrer que $l(M) = l(M') + l(M'')$)

Preuve du fait sur les chaînes : étant donné une chaîne dans M' et une chaîne dans M'', il est très facile de construire une chaîne dans M en les collant toutes les deux. Cela montre que $l(M) \geq l(M') + l(M'')$. Pour le sens inverse, prenez une chaîne dans M, $C_0 \subset \cdots \subset C_n$. On note f l'inclusion $f : M' \to M$. On trouve une chaîne $(f^{-1}(C_i))_i$ dans M' et une chaîne dans M'' obtenue par projection.

Montrez que, dès que l'image de C_i dans M'' est la même que celle de C_{i-1}, on doit avoir $f^{-1}(C_{i-1}) \subsetneq f^{-1}(C_i)$. Donc n ne peut pas excéder $l(M')+l(M'')$, puisque, pour chaque i entre 1 et n, il y a une inclusion stricte quelque part dans la chaîne dans M', ou celle dans M''.

On va montrer que $l(R)$ est fini. Notons $\mathfrak{m}$ l'idéal maximal de R et k tel que $\mathfrak{m}^k = 0$. Remarquons que la longueur d'un espace vectoriel de dimension finie est finie. Or, pour tout l entre 1 et k, on a un $R/\mathfrak{m}$-espace vectoriel $\mathfrak{m}^l/\mathfrak{m}^{l+1}$. Il est de dimension finie puisque $\mathfrak{m}^l$ est engendré par un nombre fini d'éléments (puisque A est noéthérien). Donc le $R/\mathfrak{m}$-module $\mathfrak{m}^l/\mathfrak{m}^{l+1}$ est artinien. Il est donc encore de longueur finie comme R-module.

On pose par convention $\mathfrak{m}^0 = R$. Le fait précédent est encore vrai avec $l = 0$.

On peut donc montrer par récurrence sur l que pour tout l entre 0 et k, $\mathfrak{m}^l$ est de longueur finie. Pour $l = k$, c'est trivial, puisque $\mathfrak{m}^k = 0$. Si maintenant $1 \leq l \leq k$ et $\mathfrak{m}^l$ est de longueur finie, alors on a une suite exacte $0 \to \mathfrak{m}^l \to \mathfrak{m}^{l-1} \to \mathfrak{m}^{l-1}/\mathfrak{m}^l \to 0$. Donc $\mathfrak{m}^{l-1}$ est de longueur finie comme R-module.

Donc R est un R-module de longueur finie donc artinien.

4. En déduire que A est Artinien.

Correction. A est un produit fini d'anneaux artiniens, ce n'est pas dur de voir qu'il est artinien aussi.

5. Donner un fermé de $\mathbb{A}^n_k$ dont l'anneau des fonctions est de dimension 0.

Correction. N'importe quel ensemble fini.

Exercise 3. (*Krull's Hauptidealsatz*) — Let A be a Noetherian ring and let $\mathfrak{p}$ be a prime ideal of A. The height of $\mathfrak{p}$, denoted by $\mathrm{ht}(\mathfrak{p})$ is the maximal lenght of a decreasing chain of prime ideals $\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_r$. As A is Noetherian it is really a maximal lenght, and it is finite (use exercise 1 !).

The goal of this exercise is to show the following fundamental theorem : If $f \in A \setminus (A^\times \cup \{0\})$, then any prime ideal $\mathfrak{p}$ of A containing f and minimal for this property has height 1.

1. We take f and $\mathfrak{p}$ as in the statement.

a. Show that to prove the statement, it is enough to prove it when A is local with maximal ideal $\mathfrak{p}$. In that setting, show that $\dim A/(f) = 0$.

Correction : The prime ideals of $A_\mathfrak{p}$ are in bijection with the prime ideals of A contained in $\mathfrak{p}$ hence we can assume that $A = A_\mathfrak{p}$ and $\mathfrak{p}$ is the maximal ideal of A. $\mathfrak{p}$ is still the minimal prime containing f and we still have $f \notin A^\times \cup \{0\}$.

In that case, the prime ideals of $A/(f)$ are in bijection with the prime ideals of A containing f. But the prime ideals of A are included in $\mathfrak{p}$ by locality of A. But by minimality of $\mathfrak{p}$, the only prime ideal of A/f is $\mathfrak{p}$. Hence $\dim A/(f) = 0$.

b. Let $\mathfrak{q}$ a prime ideal of A stricly contained in $\mathfrak{p}$. By assumption $f \notin \mathfrak{q}$. What do we have to show on $\mathfrak{q}$? Reduce to working in a ring A which is a domain.

Correction : We want to show that the height of $\mathfrak{p}$ is one, so that $\mathfrak{q}$ is a minimal ideal of A. It means that we want to show that there is not any prime ideal smaller than $\mathfrak{q}$ in A. Let's go further : let $\mathfrak{q}'$ be a prime ideal strictly between 0 and $\mathfrak{q}$. Then, as the ring $A/\mathfrak{q}'$ is integral its ideal 0 is prime. Hence the ideal $\mathfrak{q}/\mathfrak{q}'$ in $A/\mathfrak{q}'$ is not minimal. Hence, we are reduced to a case where A is integral and where we just want to show that $\mathfrak{q} = 0$.

c. Consider the decreasing sequence of sets $\mathfrak{q}_n := \mathfrak{q}^n A_{\mathfrak{q}} \cap A$. Show that $\forall n \geq 0, \mathfrak{q}_n = \{a \in A \mid \exists s \in A \setminus \mathfrak{q}, sa \in \mathfrak{q}^n\}$ and that it is an ideal of A.

d. Show that the sequence $(\mathfrak{q}_n \mod (f))$ of ideals of $A/(f)$ is stationnary (as $\mathfrak{q}_n$ is an ideal of A, its pushforward by the surjective ring morphism $A \to A/I$ is an ideal, denoted $\mathfrak{q}_N \mod (f)$.)

Correction : A/f is of dimension 0, hence is Artinian. Hence the sequence of ideals in A/f, $(\mathfrak{q}_n \mod (f))_n$, is stationnary.

e. Deduce that the sequence $(\mathfrak{q}_n)$ of ideals of A is stationary (use Nakayama's lemma).

Correction. This question is subtle. We know that for all N big enough, $\mathfrak{q}_N \mod (f) = \mathfrak{q}_{N+1} \mod (f)$. Pulling-back into A this relation, we get $\mathfrak{q}_N + (f) = \mathfrak{q}_{N+1} + (f)$ which simplifies to $\mathfrak{q}_N = \mathfrak{q}_{N+1} + (f)$. One could deduce that $\mathfrak{q}_N = \mathfrak{q}_{N+1} + \mathfrak{p}$, but the subtlety is that this is not enough for using Nakayama.

Let $x \in \mathfrak{q}_N$. Then there is $y \in \mathfrak{q}_{N+1}$ and $a \in A$ such that $x = y + af$. Let's show that $a \in \mathfrak{q}_N$. $af = x - y \in \mathfrak{q}_N$ so there is $s \in A - \mathfrak{q}$ such that $afs \in \mathfrak{q}^N$. But $f \notin \mathfrak{q}$ so $sf \notin \mathfrak{q}$ because $\mathfrak{q}$ is prime. So, by definition of $\mathfrak{q}_N$, $a \in \mathfrak{q}_N$.

Hence the relation becomes $\mathfrak{q}_N = \mathfrak{q}_{N+1} + (f)\mathfrak{q}_N$. As $(f) \subset \mathfrak{p}$, $\mathfrak{q}_N = \mathfrak{q}_{N+1} + \mathfrak{p}\mathfrak{q}_N$.

Hence, the finite-type module on the local ring $(A, \mathfrak{p})$, $M := \mathfrak{q}_N/\mathfrak{q}_{N+1}$, verifies $M = \mathfrak{p}M$. By Nakayama's lemma, $\mathfrak{q}_N = \mathfrak{q}_{N+1}$. This is true for all N big enough hence the sequence is stationnary.

f. Using Nakayama's lemma in $A_{\mathfrak{q}}$, show that $\mathfrak{q} = 0$.

Correction : In A, we have that, for N big enough, $\mathfrak{q}_N = \mathfrak{q}_{N+1}$. As $\mathfrak{q}_N$ is simply $\mathfrak{q}^N \cap A$, we have $\mathfrak{q}_N A_{\mathfrak{q}} = \mathfrak{q}^N A_{\mathfrak{q}}$. Hence we have $\mathfrak{q}^N A_{\mathfrak{q}} = \mathfrak{q}^{N+1} A_{\mathfrak{q}}$ for all N big enough.

The $A_{\mathfrak{q}}$-module $\mathfrak{q}^N A_{\mathfrak{q}}$ is of finite type. It is true because localizing a noetherian ring is noetherian so the ideal in $A_{\mathfrak{q}}$ generated by $\mathfrak{q}^N$ is of finite type, hence the module $\mathfrak{q}^N A_{\mathfrak{q}}$ is finitely generated.

Hence Nakayama's lemma gives that $\mathfrak{q}^N A_{\mathfrak{q}} = 0$. But A is integral so this just means that $\mathfrak{q} = 0$.

2. Let A be a local and Notherian ring. Let $I = (f_1, \ldots, f_n)$ an ideal of A. Show that $\dim A \leqslant \dim A/I + r$ (do it first for $r = 1$). Deduce that the dimension of A is less than the dimension as a $k = A/\mathfrak{m}$ vector space of $\mathfrak{m}/\mathfrak{m}^2$. In particular, the dimension of A is finite.

Correction : Let $\overline{x_1}, \ldots, \overline{x_n}$ be a basis of $\mathfrak{m}/\mathfrak{m}^2$. By Nakayama's lemma, $\mathfrak{m} = (x_1, \ldots, x_n)$ with x_i a lift of $\overline{x_i}$. Let us show by induction on r that if $f_1, \ldots, f_r \in \mathfrak{m}$, then $\dim A \leqslant \dim_k(A/(f_1, \ldots, f_r)) + r$. This will answer the question.

By induction, it suffices to show that if $f \in \mathfrak{m}$, $\dim A \leqslant \dim(A/f) + 1$. Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \mathfrak{p}_n$ a chain of prime ideals of A with $f \in \mathfrak{p}_n$. We will show by induction on n that there exists a chain of prime ideals $\mathfrak{p}'_0 \subset \mathfrak{p}'_1 \subset \cdots \subset \mathfrak{p}'_n$ such that $\mathfrak{p}_0 = \mathfrak{p}'_0$ and $f \in \mathfrak{p}'_1$.

The case $n = 1$ is trivial. For $n \geqslant 2$, we can assume $f \notin \mathfrak{p}_{n-1}$ as otherwise the induction hypothesis would trivially finish the induction. We can choose a prime ideal $\mathfrak{p}'_{n-1}$ of A that contains $\mathfrak{p}_{n-2} + (f)$ and is minimal for that property (this exists by taking an irreducible component of $\operatorname{Spec} A/\mathfrak{p}_{n-2} + (f)$.) As the height of $\mathfrak{p}_n$ is at least 2 in the ring $A/\mathfrak{p}_{n-2}$, the Krull Hauptidealsatz shows that $\mathfrak{p}'_{n-1} \subset \mathfrak{p}_n$ is strict. The induction hypothesis applied to $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{n-2} \subset \mathfrak{p}'_{n-1}$ gives a chain that when concatened with $\mathfrak{p}'_n := \mathfrak{m}$ gives the result.

We apply what we just proven to any chain of A with top ideal $\mathfrak{m}$. This shows that $\dim(A/f) \geqslant \dim A - 1$.

Remark : As said in the question, this shows that any Noetherian *local* ring is of finite dimension. There exists Noetherien rings that do not have finite Krull dimension, an example was constructed by Nagata.

Exercise 4. (*An application of the Hauptidealsatz*) —

Let X be an affine irreducible variety. Let $f \in \mathscr{O}_X(X)$ be a regular function on X which is not invertible and not zero. Show that every irreducible component of $V(f) \subset X$ has dimension $\dim X - 1$.

Correction : This is just a translation of our theorem to this geometric setting : Take $A = k[X] = \mathscr{O}_X(X)$. Then irreducible components of $V(f)$ are in bijection with minimal prime ideals of $k[V(f)]$ which are in bijction with prime ideals of $k[X]$ containing f and minimal for that property. Then by the Hauptidealsatz, the lenght of those ideals is 1, This means that for any chain of irreducible closed subset $Z_0 \subset Z_1 \subset \cdots Z_n = Z$ of an irreducible component Z of $V(f)$ with Z_n , corresponding to prime ideals $\mathfrak{p}_n \subset \mathfrak{p}_0$ of $k[X]$ not containg f, and $\mathfrak{p}_n$ minimal for that property, we can always add one, and only one, prime ideal $\mathfrak{p}_{n+1}$ of $k[X]$, strictly contained in $\mathfrak{p}_n$, so that $\dim Z = \dim X - 1$ (by starting from a maximal chain in Z we obtain a maximal chain in X.)

Exercise 5. (*Hypersurfaces, codimension* 1 *and a trap.*) —

1. Let X be an affine algebraic variety such that $\mathscr{O}_X(X)$ is a factorial ring.

a. Let $f \in \mathscr{O}_X(X)$ irreducible. Show that $\dim V(f) = \dim X - 1$ (for this you do not need the Hauptidealsatz.)

Correction : f is irreducible so that $V(f)$ is irreducible. A factorial ring is a domain so that X is irreducible. Therefore, any lenght n chain of irreducible closed subset of $V(f)$ can be enlarged to have a chain of lenght n of X : $\dim V(f) \simeq \dim X + 1$.

Conversely, let $V(f) \subset Z \subset X$ with Z an irreducible closed subset with $Z \neq X$. We will show that $V(f) = Z$, proving that $\dim V(f) = \dim X - 1$. Let $\mathfrak{p}$ be a prime ideal of $k[X]$ such that $Z = V(\mathfrak{p})$. For each $x \in \mathfrak{p}$, there is one of the irreducible elements in the prime decomposition of x that has to belong to $\mathfrak{p}$, proving that $\mathfrak{p}$

has a generating set made of irreducible elements. As $\mathfrak{p} \subset (f)$, f divides all of these elements, which is absurd unless there is only one : f. Therefore $\mathfrak{p} = (f)$ and we are done.

b. Let $Y \subset X$ an irreducible closed subset of X such that $\dim Y = \dim X - 1$. Show that there exists $f \in \mathscr{O}_X(X)$ irreducible such that $Y = V(f)$.

Correction : As in the previous question, we choose a prime ideal $\mathfrak{p}$ of $k[X]$ such that $Y = V(\mathfrak{p})$, and we take $f \in \mathfrak{p}$ an irreducible element (there is one by looking at the decompostion of a nonzero element of $\mathfrak{p}$.) By the previous question $\dim V(f) = \dim Y$. We have that $Y \subset V(f)$. If the inclusion was strict, then a maximal chain of irreducible closed subset for Y would give a chain of irreducible closed subset of $V(f)$ of lenght $\dim V(f) + 1$ which is absurd. Therefore, $Y = V(f)$.

2. Let $X = V(ad - bc) \subset \mathbb{A}^4_k$ and let $Y = V(a, b) \subset X$. Show that Y is a closed irreducible closed subset of X such that $\dim Y = \dim X - 1$. Show that there are no $f \in \mathscr{O}_X(X)$ such that $Y = V(f)$.

Correcion : Let $k[a, b, c, d] \to k[x, y]$ be the map that sends a to 0, b to 0 , c to x and d to y. This is a surjective map of rings that factors through $k[a, b, c, d]/(ac - bd) = k[V(ac - bd)]$ (we have indeed that $I(V(ac - bd))) = (ac - bd)$ because $ac - bd$ is an irreductible element of $k[a, b, c, d]$. It factors through $I(a, b) = (a, b)$ giving an isomorphism. Hence $k[V(a, b)] = k[x, y]$ is integral so that $Y = V(a, b) \simeq \mathbb{A}^2_k$ is irreducible of dimension $2 = 3 - 1 = \dim X - 1$ (the computation of $\dim X$ is by the question 1a.)

The variety X is the set of non invertible matrices 2×2 and the variety Y is the set of those matrices with zero first line. Assume there is some $f \in \mathscr{O}_X(X)$ such that $Y = V(f)$. Note that $V(a, b)$ in $V(ad - bc)$ equals $V(a, b)$ in $\mathbb{A}^n_k$ as matrices with a zero first line are non invertible. Therefore, if $g \in k[a, b, c, d]$ is a lift a f, we have that $g(a, b, c, d) = 0$ if and only if $ad - bc = 0$ and $a = b = 0$ if and only if $a = b = 0$. Therefore, $Y = V(g)$ in $\mathbb{A}^4_k$, si that, as Y is irreducible, $\dim Y = \dim X - 1$. Therefore $2 = 3$ which is not possible. Absurd, and Y cannot be defined by only one equation !