TD10 : Birational maps, tangent spaces, dimension

Exercises with a  are algebraic geometry exercices which will be corrected during the exercise session, if time allows it. Exercises with a  are important exercices of commutative algebra.

A function $f \in k[X_0, \dots, X_n, Y_0, \dots, Y_m]$ is said to be *bihomogeneous* of degrees (a, b) in the variables $X_0, \dots, X_n$ and $Y_0, \dots, Y_m$ if f is homogeneous of degree a in the X-variables and homogeneous of degree b in the Y-variables.

We recall what we got last time about the product of two projective spaces : let $V = \mathbb{P}^n_k \times \mathbb{P}^m_k$, then V can be seen as a variety (meaning that it's irreducible). Denote $X_0, \dots, X_n$ are the coordinates on $\mathbb{P}^n_k$ and $Y_0, \dots, Y_m$ are the coordinates on $\mathbb{P}^m_k$. Then $k(V)$ is exactly the field of functions of the form $\frac{f}{g}$, where f and g are bihomogeneous of the same degrees.

It is also not hard to check that rational map towards V are exactly maps $f : W \to V$ where W is a variety and such that $\pi_1 \circ f$ and $\pi_2 \circ f$ are rational, where π_1, π_2 are the canonical projections $V \to \mathbb{P}^n_k$ and $V \to \mathbb{P}^m_k$.

Scheme theory allows you to reformulate the previous paraphraphs by "fibered products exist".

Exercise 1. (*Some birational isomorphisms*) —

1. Let $U_0 = \{x_0 \neq 0\} \subset \mathbb{P}^n_k$, $V_0 = \{y_0 \neq 0\} \subset \mathbb{P}^m_k$ and $W_0 = \{z_0 \neq 0\} \subset \mathbb{P}^{n+m}_k$. Show that there is an isomorphism $U_0 \times V_0 \simeq W_0$ that you will describe in terms of projective coordinates. Deduce that $\mathbb{P}^n_k \times \mathbb{P}^m_k$ and $\mathbb{P}^{m+n}_k$ are birational.

2. Let $S = V_+(XY - ZW) \subset \mathbb{P}^3_k$, and $\Sigma := S \cap U_0 = V(Y - ZW) \subset \mathbb{A}^3_k$.

a. Show that S and Σ are birational to $\mathbb{P}^2_k$ (only use previous exercises).

b. Show that S contains two families of projective lines, each of the form $(L_t)_{t \in \mathbb{P}^1_k}$ verifying $\forall t, u, L_t \neq L_u \Rightarrow L_t \cap L_u = \emptyset$, and such that the intersection between one line from one family with one line from the other family is always a point.

c. Show that the Zariski topology on S is not the same as the product topology on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ (yes, again this kind of question).

Exercise 2. (*Singularities*) — Let k be algebraicly closed (if needed). Give the dimension, the tangent space and the singular points of V for each affine variety V given below :

1. When is the hypersurface $V_d \subset \mathbb{P}^n$ defined by $X_0^d + \cdots + X_n^d = 0$ non-singular, according to k ?

2. $V = V(Y - f(X))$ for some $f \in k[X]$.

3. V is an affine subspace of $\mathbb{A}^n_k$.

4. $V = \mathrm{GL}_n(k)$.

5. $V = \mathrm{SL}_n(k)$.

6. $V = O_n(k)$.

Définition. Let A be a commutative ring. The Krull dimension of A is the supremum of the length of all chains of proper prime ideals. (we say that a chain with only 1 element is of length 0)

Exercise 3. (*Anneaux de dimension 0*) — Soit A un anneau noetherien de dimension 0. On veut montrer que A est *artinien*, c'est à dire que toute suite décroissante d'idéaux de A stationne.

1. Montrez que A n'a qu'un nombre fini d'idéaux maximaux (qui sont aussi minimaux).

2. On note $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ les ideaux maximaux de A. Montrez qu'il existe un entier $k \in \mathbb{N}^*$ tel que A soit isomorphe au produit $\prod_i A/\mathfrak{m}^k$. Et que chacun des $A/\mathfrak{m}^k$ est local.

3. Soit R un anneau local noetherien tel dont l'idéal maximal est nilpotent. Montrez que R est Artinien

4. En déduire que A est Artinien.

5. Montrer qu'il n'y a que deux dimensions possibles pour un anneau principal.

Exercise 4. — Prove that the intersection between an hypersurface V which is not an hyperplane in $\mathbb{A}^n_k$ and $T_P V$ is singular at P (for this semester, the notion of singularity only exists for varieties, so suppose that this intersection is actually a variety, even it is not necessary at all).