ENS de Lyon
M1

Géométrie Algébrique
2025-2026

TD10 : Birational maps, tangent space, dimension

A function $f \in k[X_0, \dots, X_n, Y_0, \dots, Y_m]$ is said to be *bihomogeneous* of degrees (a, b) in the variables $X_0, \dots, X_n$ and $Y_0, \dots, Y_m$ if f is homogeneous of degree a in the X-variables and homogeneous of degree b in the Y-variables.

We recall what we got last time about the product of two projective spaces : let $V = \mathbb{P}_k^n \times \mathbb{P}_k^m$, then V can be seen as a variety (meaning that it's irreducible). Denote $X_0, \dots, X_n$ are the coordinates on $\mathbb{P}_k^n$ and $Y_0, \dots, Y_m$ are the coordinates on $\mathbb{P}_k^m$. Then $k(V)$ is exactly the field of functions of the form $\frac{f}{g}$, where f and g are bihomogeneous of the same degrees.

It is also not hard to check that rational map towards V are exactly maps $f : W \to V$ where W is a variety and such that $\pi_1 \circ f$ and $\pi_2 \circ f$ are rational, where π_1, π_2 are the canonical projections $V \to \mathbb{P}_k^n$ and $V \to \mathbb{P}_k^m$.

Scheme theory allows you to reformulate the previous paraphraphs by "fibered products exist" (not exactly cartesian products).

Exercise 1. (*Some birational isomorphisms*) —

1. Let $U_0 = \{x_0 \neq 0\} \subset \mathbb{P}_k^n$, $V_0 = \{y_0 \neq 0\} \subset \mathbb{P}_k^m$ and $W_0 = \{z_0 \neq 0\} \subset \mathbb{P}_k^{n+m}$. Show that there is an isomorphism $U_0 \times V_0 \simeq W_0$ that you will describe in terms of projective coordinates. Deduce that $\mathbb{P}_k^n \times \mathbb{P}_k^m$ and $\mathbb{P}_k^{m+n}$ are birational.

2. Let $S = V_+(XY - ZW) \subset \mathbb{P}_k^3$, and $\Sigma := S \cap U_0 = V(Y - ZW) \subset \mathbb{A}_k^3$.

a. Show that S and Σ are birational $\mathbb{P}_k^2$ (only use previous exercises).

b. Show that S contains two families of projective lines, each of the form $(L_t)_{t \in \mathbb{P}_k^1}$ verifying $\forall t, u, L_t \neq L_u \Rightarrow L_t \cap L_u = \emptyset$, and such that the intersection between one line from one family with one line from the other family is always a point.

c. Show that the Zariski topology on S is not the same as the product topology on $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ (yes, again this kind of question).

Correction. 1. Define $f(x_0, \dots, x_n, y_0, \dots, y_m) = (1 : \frac{x_1}{x_0} : \dots : \frac{x_n}{x_0} : \frac{y_1}{y_0} : \dots : \frac{y_m}{y_0})$ using previous exercise. f is the birational map you need.

2. a. this is not only birational but isomorphic by definition.

b. $\mathbb{P}^1 \times \mathbb{P}^1$ is the answser.

c. the line $x = y$ is closed S but show it is not in $\mathbb{P}^1 \times \mathbb{P}^1$. If it were closed, call it V, and then for all $x \notin V$, there is some elementary closed subset $V(f) \times \mathbb{P}^1$ or $\mathbb{P}^1 \times V(f)$ for f homogeneous that contains V and that does not contain x. Show that such an f should be constant.

Exercise 2. (*Singularities*) — Give the dimension, the tangent space and the singular points of V for each affine variety V given below :

1. When is the hypersurface $V_d \subset \mathbb{P}^n$ defined by $X_0^d + \cdots + X_n^d = 0$ non-singular, according to k ?

2. $V = V(Y - f(X))$ for some $f \in k[X]$.

3. V is an affine subspace of $\mathbb{A}_k^n$.

4. $V = \mathrm{GL}_n(k)$.

5. $V = \mathrm{SL}_n(k)$.

6. $V = O_n(k)$.

Correction. 1. Call the equation f. The question is wether or not there is a point $P = (p_0, \ldots, p_n)$ (suppose $p_0 = 1$ and work in the affine chart given by $p_0 \neq 0$) such that $f(P) = 0$ and $\forall j > 0$, $\frac{\partial f}{\partial x_j}(P) = 0$. In characteristic not dividing d, there is no such point (easy). Otherwise, the point $(1 : \ldots : 1)$ is always singular.

2. For all point $P \in V$, $T_P V$ is the line through P of direction $f'(P)$. Hence V is smooth everywhere and of rank 1.

3. V can be written $V = V(f_1, \ldots, f_r)$ where each f_k can be written $f_k = \tilde{f}_k + f_k(0)$ where $\tilde{f}_k$ is linear. Let $P \in V$. Then for all i, $\frac{\partial f_k}{\partial x_i}$ is the coefficient of f_k for x_i so $\sum_i \frac{\partial f_k}{\partial x_i} x_i = \tilde{f}_k$. The tangent space is given by the equations $\sum_i \frac{\partial f_k}{\partial x_i}(X_i - p_i) = 0$ (where $P = (p_1, \ldots, p_n)$). Those equations are then just $\tilde{f}_k - \tilde{f}_k(P) = 0$. But using the relation $f_k = \tilde{f}_k + f_k(0)$, one has $\tilde{f}_k - \tilde{f}_k(P) = f_k - f_k(P) = f_k$. Hence $T_P V = V$.

4. Let $f = Z\det(X_{ij}) - 1$. Then $\frac{\partial f}{\partial z} = \det(X_{ij})$ and $\frac{\partial f}{\partial X_{ij}} = Z\tilde{X}^{i,j}$ where $\tilde{X}^{i,j}$ is the (i,j)-th cofactor of (X_{ij}) (here we really don't care about what it is : it is just some polynomial in Z, X_{ij}, and those variables will be replaced by $1/\det(M)$ and M_{ij}). If $P \in \mathrm{GL}_n(k)$, then $T_P V$ is given by the single equation $Z\det(P) - 1 + (\ldots) = 0$ where $(\ldots)$ only depends on M and affinely on the variables X_{ij}. Hence $T_P V$ is clearly in an affine bijection with $\mathcal{M}_n(k)$ hence $T_P V$ has dimension n^2 (as an affine space). It follows that $\mathrm{GL}_n(k)$ is of dimension n^2 and smooth.

5. Here, $f = \det(X_{ij}) - 1$, so $T_P V$ is given by $\sum_{ij}(X_{ij} - P_{ij})\tilde{P}^{ij} = 0$. Think geometrically : what is the differencial of the determinant ? We have "$d\det_P(H)$" $= \sum_{ij} \frac{\partial \det}{\partial X_{ij}}(P) H_{ij}$ (this is true in differential calculus but this makes no sense in algebraic geometry). We also have (remember from differential calculus) that "$d\det_P(H) = \mathrm{tr}(P^{-1}H)$.

Let's simply prove algebraicly that for every $A \in \mathrm{GL}_n(k)$ and $H \in \mathcal{M}_n(k)$, $\mathrm{tr}(A^{-1}H) = \sum_{ij} \frac{\partial \det}{\partial X_{ij}} H_{ij}$. Both expressions are linear in H so let's just prove it for H being an elementary matrix E_{ij}. Then $\mathrm{tr}(A^{-1}H) = (A^{-1})_{ij}$ is the (i,j)-th cofactor of A and we saw that it is $\frac{\partial \det}{\partial X_{ij}}(A)$.

Hence $\sum_{ij}(X_{ij} - P_{ij})\tilde{P}^{ij} = \mathrm{tr}(P^{-1}((X_{ij}) - (P_{ij})))$. Hence we are done proving that $T_P(V)$ is the set of H such that $\mathrm{tr}(P^{-1}H) = n$. Hence $T_P(V)$ is affinely the same as the set of all H such that $\mathrm{tr}(P^{-1}H) = 0$. This vector space is of codimension 1 because the linear form $H \mapsto \mathrm{tr}(P^{-1}H)$ is non-zero because $P^{-1} \neq 0$. Hence $\mathrm{SL}_n(k)$ is of dimension $n^2 - 1$ and smooth.

6. $O_n(k)$ is the set of matrices P such that $PP^T = I_n$. Let $\phi(P) = PP^T$ and ϕ^{kl} the coordinates of ϕ (I know that k is already denoting the field but I want to denote those coordinates k, l, and there is no ambiguity with the context). If $P \in O_n(k)$

then T_PV is (up to some translation) the set of matrices H such that

$$\forall k, l, \ \sum_{ij} \frac{\partial \phi^{kl}}{\partial X_{ij}}(P) H_{ij} = 0.$$

(this is true because V is defined as the set of the P such that $PP^T = I_n$ i.e. such that $\forall k, l, \phi^{kl}(P) = \delta_{kl}$, where δ_{kl} is 1 iff $k = l$, else 0 (Kronecker's symbol))

(notice that PP^T is symmetric so there is some redundancy in those equations (only $n(n+1)/2$ are meaningul) but we don't care about that for now)

From differential calculus we recognize $d\phi_P(H)$, which is exactly $HP^T + (HP^T)^T$. Lets show that

$$\forall P \in O_n(k), H \in \mathcal{M}_n(k), k, l, \sum_{ij} \frac{\partial \phi^{kl}}{\partial X_{ij}}(P) H_{ij} = (HP^T + (HP^T)^T)_{kl}.$$

Look at the coordinates : it suffices to show that

$$\forall P \in O_n(k), i, j, k, l, \frac{\partial \phi^{kl}}{\partial X_{ij}}(P) = (E_{ij}P^T + (E_{ij}P^T)^T)_{kl},$$

where E_{ij} is the elementary matrix with only a 1 on the i-row and on the j-column.

In one hand, simple computations lead to $E_{ij}P^T = \begin{pmatrix} 0 \\ - \\ p_{1j} \cdots p_{nj} \\ - \\ 0 \end{pmatrix}$ where the line is in i-th position. Hence $(E_{ij}P^T + (E_{ij}P^T)^T)_{kl} = p_{lj}\delta_{ki} + p_{kj}\delta_{li}$.

On the other hand, $\frac{\partial \phi^{kl}}{\partial X_{ij}}(P)$ is the same because $\phi^{kl}(x) = \sum_{u=1}^{n} x_{ku}x_{lu}$ and the derivative of such a thing is what we want (discuss the 4 cases).

Hence we proved our formula, hence T_PV is the set of H such that $HP^T + (HP^T)^T = 0$. This is always of dimension of the set of antisymmetric matrices, $n(n-1)/2$.

Exercise 3. — Prove that the intersection between an hypersurface V which is not an hyperplane in $\mathbb{A}_k^n$ and T_PV is singular at P (for this semester, the notion of singularity only exists for varieties, so suppose that this intersection is actually a variety).

Correction. Denote W the intersection $V \cap T_PV$. $V \neq T_PV$ so by the Hauptidealsatz $\dim(W) < \dim(V)$. We also have $\dim(V) = n - 1$. Denote $f = f_0 + f_1 + f_2 + \cdots + f_r$ the Taylor expansion of f. We can translate P so that $P = 0$ hence $f_0 = 0$. f_1 is the equation of T_PV so $W = (f, f_1) = (f_2 + \cdots + f_r, f_1)$. $f_2 + \cdots + f_r$ has no linear term so T_PW is defined by one equation, $\nabla f_1 \cdot X = 0$ (∇f_1 is the gradient of f_1), so it is in fact an hypersurface, of dimension at least $n - 1$. Hence $dim_k(T_PW) > rank(W)$, hence P is singular.

Définition. Let A be a commutative ring. The Krull dimension of A is the supremum of the length of all chains of proper prime ideals. (we say that a chain with only 1 element is of length 0)

Exercise 4. (*Anneaux de dimension 0*) — Soit A un anneau noetherien de dimension 0. On veut montrer que A est *artinien*, c'est à dire que toute suite décroissante d'idéaux de A stationne.

1. Montrez que A n'a qu'un nombre fini d'idéaux maximaux (qui sont aussi minimaux).

2. On note $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ les ideaux maximaux de A. Montrez qu'il existe un entier $k \in \mathbb{N}^*$ tel que A soit isomorphe au produit $\prod_i A/\mathfrak{m}^k$. Et que chacun des $A/\mathfrak{m}^k$ est local.

3. Soit R un anneau local noetherien tel dont l'idéal maximal est nilpotent. Montrez que R est Artinien

4. En déduire que A est Artinien.

5. Montrer qu'il n'y a que deux dimensions possibles pour un anneau principal.