ENS de Lyon
M1

Géométrie Algébrique
2025-2026

TD1 : The Nullstellensatz

Exercises with a  are algebraic geometry exercices which will be corrected during the exercise session, if time allows it. Exercises with a are important exercices of commutative algebra.

Algebraic geometry is the study of systems polynomial equations by using geometric ideas.

Today's exercice session is devoted to a technical warm-up in commutative algebra, with a view towards the Nullstellensatz and a beautiful result of algebraic geometry :

Theorem (Ax-Grothendieck) : Let k an algebraicly closed field and $P : k^n \to k^n$ a polynomial map (i.e. there are polynomials in n variables $P_1, \ldots, P_n$ such that $\forall x \in k^n, P(x) = (P_1(x), \ldots, P_n(x))$).

$$\text{Then : } P \text{ injective} \Rightarrow P \text{ surjective.}$$

This theorem can be proved for $k = \mathbb{C}$ using complex analysis. For the general case, We will need a good chunk of commutative algebra.

Some crucial definitions to begin with : let k a field. Let I an ideal of $k[X_1, \ldots, X_n]$. I can be thought as a system of equations and we can define

$$V(I) := \{x \in k^n \mid \forall f \in I, f(x) = 0\},$$

called *closed* subsets of k^n (you will see why later). For example : the circle in $\mathbb{R}^2$ is the closed subset $V((X^2 + Y^2 - 1))$.

If $V \subset k^n$, we can try to recover an ideal of equations : the functions which vanish on V : $I(V) := \{f \in k[X_1, \ldots, X_n] \mid \forall x \in V, f(x) = 0\}$.

Problem. Let V be a closed subset. You will learn soon that it is easy to see that $V(I(V)) = V$. Do we have the converse, i.e., if I is an ideal, do we have $I = I(V(I))$? Some straightforward calculation show that $I(V(I))$ contains not only I but also $\sqrt{I} := \{x \in k[X_1, \ldots, X_n] \mid \exists n \in \mathbb{N}, x^n \in I\}$. And there is nothing else :

Exercise 1. (*The Nullstellensatz*) — Let K be an algebraicly closed field. Recall the weak form of the nullstellensatz : the maximal ideals of $K[X_1, \ldots, X_n]$ are those of the form $(X_1 - a_1, \ldots, X_n - a_n)$, with $a_1, \ldots, a_n \in K$. Let's look for some corollaries.

1. Let $P_1, \ldots, P_r$ be elements of $K[X_1, \ldots, X_n]$ that do not have any common root in K^n. Can they have a common root in L^n, where L is some field extension of K ?

2. Let I be an ideal of $K[X_1, \ldots, X_n]$, denote $V = V(I)$.

a. Show that $\sqrt{I}$ is an ideal of $K[X_1, \ldots, X_n]$.

b. Show that $\sqrt{I} \subset I(V)$.

c. Let $P \in I(V)$. Let J be the ideal of $K[X_1, \ldots, X_n, T]$ generated by I and $1 - PT$. Prove that $J = K[X_1, \ldots, X_n, T]$.

d. Deduce that $I(V) = \sqrt{I}$ (Hilbertscher Nullstellensatz).

Exercise 2. (*Artin-Tate lemma and applications*) — **1.** (Artin-Tate) Let $A \subset B \subset C$ be rings with A Noetherian, C of finite type as an A-algebra and of finite type as a B-module. Show that B is a finite type A-algebra. *(First, show the existence of $x_1, \ldots, x_n$ in B such that C is a finite type D-module, with $D = A[x_1, \ldots, x_n]$.)*

2. (Zariksi) Let $K \subset L$ be a field extension. Show that if L is a finite type K-algebra, then L/K is a finite extension.

a. Show that there exists a family $x_1, \ldots, x_n$ of mutually algebraicly independant elements of L such that L is finite over $K(x_1, \ldots, x_n)$. Conclude.

b. (subsidiary question) Can $K(X_1, \ldots, X_n)$ be finite over $K[X_1, \ldots, X_n]$?

3. (Jacobson) Let K be a field and let A be a finite type K-algebra. Show that if $\mathfrak{m}$ is a maximal ideal of A, then $A/\mathfrak{m}$ is a finite extension of K. In the same vein, if A is a finite type $\mathbb{Z}$-algebra and $\mathfrak{m}$ is a maximal ideal of A, then $A/\mathfrak{m}$ is a finite field.

4. (Hilbert-Noether) Let K be a field, A a finite type K-algebra and G a finite group acting on A by K-algebra homomorphisms. Show that

$$A^G := \{a \in A \mid g \cdot a = a, \ \forall g \in G\}$$

if a finite type K-algebra.

Exercise 3. (*Nilradical of a ring*) — Let A be a commutative ring. Recall the following definition : A partially ordered set $\mathcal{P}$ is *inductive* if any totally ordered subset $\mathcal{T} \subset \mathcal{P}$ has a supremum : $\min\{q \in \mathcal{P} \mid \forall t \in \mathcal{T}, t \leqslant q\}$ exists. We will use the following lemma (equivalent to the axiom of choice) in this exercise :

Zorn's Lemma

Any inductive partially ordered set has a maximal element.

1. Let $S \subset A$ be a multiplicative subset that does not contains 0. Show that there exist an ideal $\mathfrak{p}$ of A which is maximal among ideals I of A not containing any element of S.

2. Show that any ideal $\mathfrak{p}$ as in the previous question is a prime ideal.

3. (Krull's Theorem) Denote Spec$A$ the set of prime ideals of A. Show that

$$\bigcap_{\mathfrak{p}\in \mathrm{Spec}A} \mathfrak{p} = \sqrt{(0)}.$$

4. Let $I \subset A$ an ideal. Denote $V(I) \subset \mathrm{Spec}(A)$ the set of prime ideals containing I. Prove that we have

$$\sqrt{I} = \bigcap_{\mathfrak{p}\in V(I)} \mathfrak{p}.$$

5. Let A be a reduced ring, that is a ring without nilpotent. Show that there is an injection of rings from A to a product of domains. Show it can be embedded in a product of fields.

6. (harder) Can you show that if A is reduced and Noetherian, it embeds in a finite product of fields ?

7. If A is a Noetherian ring, can you show Krull's theorem without Zorn's lemma ?

8. Relate the two notions of $V(I)$ we saw in this sheet.

Exercise 4. (*Jacobson rings*) — Let A be a commutative ring and $I \subset A$ an ideal. The *Jacobson radical* $J(I)$ of I is defined as the intersection of all the maximal ideals of A containing I.

First manipulations.

1. Show that $J((0)) = \{x \in A \mid \forall y \in A, 1 - xy \in A^\times\}$.
A is called a *Jacobson ring* if for all ideal I of A, $J(I) = \sqrt{I}$.

2. Show that the following rings are Jacobson : $\mathbb{Z}$, fields, and the rings $k[T]$ where k is a field.

3. Show that the quotient of a Jacobson ring is again Jacobson.

4. Show that a ring is Jacobson if and only if for all prime ideal $\mathfrak{p}$, $\mathfrak{p} = J(\mathfrak{p})$.

5. Show that $A[T]$ being Jacobson implies A being Jacobson.

Polynomial rings over Jacobson rings.

6. (prelude) Let A be an integral domain, $\mathfrak{p}$ a non zero prime ideal of $A[T]$.

a. Let $f, g \in A[T]$, f of non-zero leading coefficient a. Show that there exists $n \in \mathbb{N}$, $q, r \in A[T]$ with $r = 0$ or $\deg(r) < \deg(f)$, such that $a^n g = qf + r$.

b. Let $f \in \mathfrak{p} - \{0\}$ of minimal degree and a its leading coefficient. Show that $\forall h \in \mathfrak{p}, \exists n \in \mathbb{N}, a^n h \in fA[T]$.

c. If $g \in A[T] - \mathfrak{p}$, prove that $(\mathfrak{p} + (g)) \cap A \neq 0$.

d. Let $\mathfrak{m}_0$ a maximal ideal of A not containing a. Show that $a \notin \mathfrak{m}_0 A[T] + \mathfrak{p}$.

7. Let A a Jacobson ring. Let's show that $A[T]$ is Jacobson also.

a. Suppose that A is integral. Let $\mathfrak{p}$ a non zero prime ideal in $A[T]$ that intersects A trivially. Let $g \in J(\mathfrak{p}) - \mathfrak{p}$. Show that there exists $b \in (\mathfrak{p} + (g)) \cap A - \mathfrak{p}$. Show that there exists some $\mathfrak{m}_0$ maximal in A such that $a \notin \mathfrak{m}_0 A[T] + \mathfrak{p}$.

b. Let $\mathfrak{m}$ a maximal ideal of $A[T]$ containing $\mathfrak{m}_0 A[T] + \mathfrak{p}$. Show that $g \in \mathfrak{m}$ and deduce a contradiction.

c. Deduce that for all prime ideals $\mathfrak{p}$ in $A[T]$ such that $A \cap \mathfrak{p} = 0$, we have

$\mathfrak{p} = J(\mathfrak{p})$.

d. Now, let A be a general Jacobson ring, prove that $A[T]$ is Jacobson.

Generalized Nullstellensatz.

8. Let A be an integral domain. Let $\mathfrak{m}$ be maximal in $A[T]$. Suppose that $A \cap \mathfrak{m} = 0$. Let f be of minimal degree in $\mathfrak{m}$, of leading coefficient $a \in A$.

a. Let $\mathfrak{m}_0$ be any non zero maximal ideal in A. Let $b \in \mathfrak{m}_0 - \{0\}$. Explain why there exist $g \in A[T]$, $h \in \mathfrak{m}$ such that $1 = h + gb$.

b. Apply Euclidean division of some $a^n g$ by f, denote the remainder r. Show that $a^n - br \in \mathfrak{m}$. Deduce that $a \in \mathfrak{m}_0$.

9. Let A be an integral domain which is Jacobson. Show that A is a field if and only if there exists a maximal ideal $\mathfrak{m}$ of $A[T]$ such that $\mathfrak{m} \cap A = 0$.

10. Show that the following are equivalent :

(1). A is Jacobson

(2). For all maximal ideal $\mathfrak{m}$ of $A[T]$, $\mathfrak{m} \cap A$ is a maximal ideal in A.

11. (Generalized Nullstellensatz) Let A a Jacobson ring and B a commutative A-algebra of finite type.

a. Show that B is a Jacobson ring.

b. If $\mathfrak{m}$ is a maximal ideal of B, show that $A \cap \mathfrak{m}$ is maximal.

c. Show furthermore that

$$A/A \cap \mathfrak{m} \to B/\mathfrak{m}$$

is a finite extension of fields.

12. How is this theorem related to the Hilbertscher Nullstellensatz ?